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Harmonic analysis/Ordinary differential equations

Solutions of a class of multiplicatively advanced differential equations

Solutions d'une classe d'équations différentielles multiplicativement avancées

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ABSTRACT

The multiplicatively advanced differential equations (MADEs) of form $f^{(n)}(t) = \alpha f(\beta t)$ with $\alpha \neq 0$, $\beta > 1$ are studied along with a class of their solutions of type $f_{\mu,\lambda}(t)$ defined on $[0, \infty)$. For $\lambda \in \mathbb{Q}^+$, $\mu \in \mathbb{R}$, the solutions $f_{\mu,\lambda}(t)$ are extended to $(-\infty, \infty)$ in a non-unique manner to obtain Schwartz wavelet solutions $F_{\mu,\lambda}(t)$ of the original MADE, with all moments of $F_{\mu,\lambda}(t)$ vanishing. Examples are studied in detail. The Fourier transform of each $F_{\mu,\lambda}(t)$ is computed and, in a number of examples, is related to the Jacobi theta function. Additional conditions sufficient for the uniqueness of certain MADE initial value problems are given. Conditions for decay and non-decay at $-\infty$ are obtained. Decay rates at $\pm\infty$ in terms of familiar functions are established.

RÉSUMÉ

Des équations différentielles multiplicativement avancées (MADE) de la forme $f^{(n)}(t) = \alpha f(\beta t)$ avec $\alpha \neq 0$, $\beta > 1$ sont étudiées dans le cadre des solutions de type $f_{\mu,\lambda}(t)$ définies sur $[0, \infty)$. Pour $\lambda \in \mathbb{Q}^+$, $\mu \in \mathbb{R}$, les solutions $f_{\mu,\lambda}(t)$ sont prolongées sur $(-\infty, \infty)$ d'une manière non unique pour obtenir des solutions ondelettes dans l'espace de Schwartz $F_{\mu,\lambda}(t)$ de l'originale MADE, avec tous les moments de $F_{\mu,\lambda}(t)$ nuls. Des exemples sont étudiés en détail. La transformée de Fourier de chaque $F_{\mu,\lambda}(t)$ est calculée et, dans un certain nombre d'exemples, est liée à la fonction thêta de Jacobi. Des conditions supplémentaires suffisantes pour l'unicité de la solution de certaines MADE avec condition initiale sont données. Les conditions de décroissance et de non-décroissance à $-\infty$ sont obtenues. Les taux de décroissance à $\pm\infty$ en termes de fonctions familières sont établis.

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1. Introduction: multiplicatively advanced differential equations (MADEs)

This article is a study of homogeneous multiplicatively advanced differential equations (MADEs) of the form

$$f^{(n)}(t) = \alpha f(\beta t) \quad \text{or equivalently} \quad f^{(n)}(t) - \alpha f(\beta t) = 0, \tag{1}$$

where $\alpha \neq 0$ and $\beta > 1$. Note that the argument βt in the second term of (1) is multiplicatively advanced by the advancing parameter $\beta > 1$, making (1) a MADE. We shall approach this study through the examination of a new class of functions $f_{\mu,\lambda}(t)$ given in the following definition.

Definition 1.1. Let $q > 1$, $\mu \in \mathbb{R}$, and $\lambda > 0$. Then for $t \geq 0$, the function $f_{\mu,\lambda}(t)$ is given by

$$f_{\mu,\lambda}(t) \equiv \sum_{m=-\infty}^{\infty} (-1)^m \frac{\mathrm{e}^{-q^m t}}{q^{m(m-\mu)/\lambda}} . \tag{2}$$

From (2), observe that $|f_{\mu,\lambda}(t)| \leq \sum_{m=-\infty}^{\infty} 1/q^{m(m-\mu)/\lambda} < \infty$. So $f_{\mu,\lambda}(t)$ is bounded and converging uniformly on $t \in [0,\infty)$. For λ rational, the $f_{\mu,\lambda}(t)$ satisfy the MADE (18) below, which by choice of parameters involved can be shown to be equivalent to (1). This equivalence is shown in the Remark 7 following Theorem 2.2 below. Note that if one complexifies the argument t to obtain the complex argument z in (2), then the above bound $|f_{\mu,\lambda}(z)| \leq \sum_{m=-\infty}^{\infty} 1/q^{m(m-\mu)/\lambda} < \infty$ still holds for z in the right half-plane $\mathcal{R}(z) \geq 0$. As the uniform limit of the analytic functions given by the truncated summations (for m ranging from $-N$ to N in (2) as $N \to \infty$), $f_{\mu,\lambda}(z)$ is analytic [28] on the open right half-plane $\mathcal{R}(z) > 0$. Thus $f_{\mu,\lambda}(t)$ is real analytic in t on $(0,\infty)$ and it is $\mathcal{C}^\infty$ in t on $[0,\infty)$. For these values of t, it is also real analytic in the parameters μ and λ, but only $\mathcal{C}^\infty$ in the parameter $q > 1$.

After obtaining the properties of $f_{\mu,\lambda}(t)$ on the positive half-line $[0,\infty)$ including the MADE that each solves, the $f_{\mu,\lambda}(t)$ are extended to $F_{\mu,\lambda}(t)$ globally defined and $\mathcal{C}^\infty$ on all of the real line. These $F_{\mu,\lambda}(t)$ also satisfy the original MADE satisfied by $f_{\mu,\lambda}(t)$. The extensions $F_{\mu,\lambda}(t)$ are shown to be decaying rapidly at $\pm\infty$ and are in fact Schwartz wavelet functions on $\mathbb{R}$. As decaying global functions, they are amenable to Fourier transform computations, which are obtained and seen to be related to the Jacobi theta function in a range of cases. Ultimately, this study expands the connection between global solutions of MADEs such as (1) with the harmonic analysis of Schwartz wavelets, which, in turn, can be connected with the special function theory of the Jacobi theta function. As a first such connection, we point out that the formal MacLaurin series for $f_{\mu,\lambda}(t)$ is given by

$$\sum_{n\geq 0} \frac{f_{\mu,\lambda}^{(n)}(0)}{n!} z^n = \sum_{n\geq 0} \frac{(-1)^n \theta(q^{2/\lambda}; -q^{(\mu+n\lambda-1)/\lambda})}{n!} z^n , \tag{3}$$

where $\theta(q;u)$ is the Jacobi theta function given by (22) below, and where equality in (3) follows from (12) and (28). As will be seen in general in the proof of Proposition 2.3 below, the formal MacLaurin series given by (3) has radius of convergence 0 when $f_{\mu,\lambda}(t)$ is not flat at $t = 0$. Hence, $f_{\mu,\lambda}(t)$ and its extension $F_{\mu,\lambda}(t)$ cannot be real analytic at $t = 0$. Thus, in this study we restrict $F_{\mu,\lambda}(t)$ to t on the real line in the $\mathcal{C}^\infty(\mathbb{R})$ case, as opposed to attempting to extend the $f_{\mu,\lambda}(z)$ analytically beyond the imaginary axis in the complex plane, which in many cases is problematic via the Remark 4 at the end of this section. In special cases, methods of extending exponential series beyond a natural boundary, such as the imaginary axis encountered in (2), are well studied, see for instance [5]. Also, restriction of $f_{\mu,\lambda}(z)$ to the imaginary axis $z = it$ yields an almost periodic function of t, as per p. 289 of [1], see also [2], [3].

While the MADE (1) may at first appear counter-intuitive, its solutions for special values of μ and λ are generating a number of interesting applications. These special case applications include: modeling tsunami waves [25]; modeling rogue waves [25]; obtaining Schwartz functions ${}_qCos(t)$ and ${}_qSin(t)$ which well-approximate $\cos(t)$ and $\sin(t)$, respectively, on compact sets as $q \to 1^+$ [24], as illustrated in Fig. 1; obtaining smooth Schwartz approximations of the Haar wavelet [27]; obtaining Schwartz approximations of truncated Legendre polynomials [26], [27]; and obtaining Schwartz approximations of spherical Bessel functions of the first kind [27]. The majority of these solutions also turn out to be Schwartz wavelets generating wavelet frames for $\mathcal{L}^2(\mathbb{R})$, and in turn these solutions comprise a rich set within each $\mathcal{L}^p(\mathbb{R})$ space and have good decay and localization while satisfying perturbations of classic differential equations (see Remark 8 after Theorem 2.2). The solutions of (1) will also provide further interesting applications to physics. Each of the solutions described in the applications above relate to special function theory in the sense that all of them have Fourier transforms that can be expressed in terms of the Jacobi theta function (see (22) below). A pattern is emerging that clarifies the relation of solutions of MADEs such as (1) to: wavelets and wavelet frames, special function theory, approximation theory, self-similarity, and physical applications.

Thus the MADE (1) and the functions (2) deserve study in their own right. We note that Definition 1.1 is motivated by and generalizes: (i) the results in [22], where the mother wavelet $K(t) = f_{-1,2}(t)$ was shown to satisfy the MADE $K'(t) = K(qt)$