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Functional analysis

A norm inequality for positive block matrices

Une inégalité de norme pour les matrices positives écrites par blocs

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ABSTRACT

Any positive matrix $M=(M_{i,j})_{i,j=1}^m$ with each block $M_{i,j}$ square satisfies the symmetric norm inequality $\|M\|\le\|\sum_{i=1}^m M_{i,i}+\sum_{i=1}^{m-1}\omega_i I\|$, where ω_i $(i=1,\dots,m-1)$ are quantities involving the width of numerical ranges. This extends the main theorem of Bourin and Mhanna (2017) [4] to a higher number of blocks.

RÉSUMÉ

Toute matrice positive $M=(M_{i,j})_{i,j=1}^m$ écrite en blocs carrés $M_{i,j}$ satisfait $\|M\|\le\|\sum_{i=1}^m M_{i,i}+\sum_{i=1}^{m-1}\omega_i I\|$, où les quantités ω_i, $i=1,\dots,m-1$, font intervenir la largeur du domaine des valeurs numériques. Ceci étend le théorème principal de Bourin, Mhanna (2017) [4] aux matrices écrites avec un nombre de blocs arbitraire.

1. Introduction

Bourin and Mhanna recently obtained a novel norm inequality for positive block matrices.

Theorem 1.1. *[4] Let* $M=\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{2,2}\end{pmatrix}$ *be a positive matrix with each block square. Then for all symmetric norms*

$$\|M\|\le\|M_{1,1}+M_{2,2}+\omega I\|,$$

where ω *is the width of the numerical range of* $M_{1,2}$.

The numerical range (or the field of values [7]) is a convex set on the complex plane. By the width of a numerical range, we mean the smallest possible ω such that the numerical range is contained in a strip of width ω. In particular, if the

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numerical range of $M_{1,2}$ is a line segment (this happens, for example, when $M_{1,2}$ is Hermitian or skew-Hermitian), then the previous theorem gives (see [8, Theorem 2.6])

$$\|M\| \le \|M_{1,1} + M_{2,2}\|. \tag{1}$$

To the author's best knowledge, Mhanna's study [8] provides the first example for (1) to be true without the PPT (i.e. positive partial transpose) condition. We refer to [6,2] for some motivational background.

Bourin and Mhanna's proof of Theorem 1.1 makes use of a useful decomposition for 2×2 positive block matrices [1, Lemma 3.4]. Their approach seems difficult for an extension to a higher number of blocks, as remarked in their paper [4]. It is the purpose of the present paper to provide such an extension. Before closing this section, we fix some notation. The set of $m \times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}$ and we use $\mathbb{M}_n$ for $\mathbb{M}_{n\times n}$. The $n \times n$ identity matrix is denoted by I. The Hermitian part of $A \in \mathbb{M}_n$ is $\Re A := (A + A^*)/2$. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \ge B$ to mean $A - B$ is positive semidefinite. The numerical range of A is denoted by $W(A)$. If $A, B \in \mathbb{M}_n$, then we write $W(A \pm B)$ to mean $W(A + B)$ and $W(A - B)$. It is useful to notice that if the width of $W(A)$ is ω, then one can find a $\theta \in [0, \pi]$ such that

$$rI \le \Re(e^{i\theta} A) \le (r + \omega)I$$

for some $r \in \mathbb{R}$. We refer the reader to Chapter 1 of [7] for basic properties of the numerical range for matrices.

2. Main result

Our extension of Theorem 1.1 to a higher number of blocks is as follows.

Theorem 2.1. *Let $M = (M_{i,j})_{i,j=1}^m$ be a positive matrix with each block $M_{i,j} \in \mathbb{M}_n$. Then for all symmetric norms,*

$$\|M\| \le \|\sum_{i=1}^m M_{i,i} + \sum_{i=1}^{m-1} \omega_i I\|,$$

where ω_i ($i = 1, \ldots, m-1$) is the average of the widths of $W(M_{i,i+1} \pm M_{i,i+2} \pm \cdots \pm M_{i,m})$.

Proof. By Fan's dominance theorem [7, p. 206], it suffices to show that the inequality is true for the Ky Fan norms $\|\cdot\|_k$, $k = 1, \ldots, n$. The proof is by induction. The base case $m = 2$, i.e. Theorem 1.1 was treated in [4]. We include a proof for completeness. The presentation is slightly different from that in [4]. As M is positive, we may write $M = \begin{pmatrix} X^* \\ Y^* \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix}$ for some $X, Y \in \mathbb{M}_{2n\times n}$ so that $M_{1,1} = X^*X$, $M_{1,2} = X^*Y$, $M_{2,2} = Y^*Y$. Clearly, $\|M\|_k = \|XX^* + YY^*\|_k$. As the norm of M is invariant if we replace Y with $e^{i\theta}Y$, we may assume that $rI \le \Re(X^*Y) \le (r + \omega)I$ for some $r \in \mathbb{R}$ and that ω is the width of $W(M_{1,2})$. Compute

$$\begin{aligned}
\|M\|_k &= \frac{1}{2}\|(X+Y)(X+Y)^* + (X-Y)(X-Y)^*\|_k \\
&\le \frac{1}{2}\Big(\|(X+Y)(X+Y)^*\|_k + \|(X-Y)(X-Y)^*\|_k\Big) \\
&= \frac{1}{2}\Big(\|(X+Y)^*(X+Y)\|_k + \|(X-Y)^*(X-Y)\|_k\Big) \\
&\le \frac{1}{2}\Big(\|X^*X + Y^*Y + 2(r+\omega)I\|_k + \|X^*X + Y^*Y - 2rI\|_k\Big) \\
&= \|X^*X + Y^*Y + \omega I\|_k = \|M_{1,1} + M_{2,2} + \omega I\|_k.
\end{aligned}$$

This completes the proof of the base case. Suppose the asserted inequality is true for $m = \ell$ for some $\ell > 2$. Then we consider the $m = \ell + 1$ case. In this case, M could be written in the form

$$M = \begin{pmatrix} X_1^*X_1 & \cdots & X_1^*X_\ell & X_1^*X_{\ell+1} \\ \vdots & & \vdots & \vdots \\ X_\ell^*X_1 & \cdots & X_\ell^*X_\ell & X_\ell^*X_{\ell+1} \\ X_{\ell+1}^*X_1 & \cdots & X_{\ell+1}^*X_\ell & X_{\ell+1}^*X_{\ell+1} \end{pmatrix} = \begin{pmatrix} X_1^* \\ \vdots \\ X_\ell^* \\ X_{\ell+1}^* \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_\ell & X_{\ell+1} \end{pmatrix}$$

for some $X, Y \in \mathbb{M}_{(\ell+1)n\times n}$. Again, we assume (by multiplying $X_{\ell+1}$ with a rotation unit) that

$$sI \le \Re(X_\ell^* X_{\ell+1}) \le (s + \omega_\ell)I \tag{2}$$

for some $s \in \mathbb{R}$.