C. R. Acad. Sci. Paris, Ser. I 356 (2018) 823–829

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Differential geometry

Conformally flat real hypersurfaces in nonflat complex planes

Hypersurfaces réelles conformément plates dans les plans complexes non plats

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ARTICLE INFO

Article history:
Received 1 February 2018
Accepted after revision 30 April 2018
Available online 22 May 2018

Presented by the Editorial Board

ABSTRACT

In this paper, we prove that there are no conformally flat real hypersurfaces in nonflat complex space forms of complex dimension two provided that the structure vector field is an eigenvector field of the Ricci operator. This extends some recent results by Cho (Conformally flat normal almost contact 3-manifolds, Honam Math. J. 38 (2016) 59–69) and Kon (3-dimensional real hypersurfaces with η-harmonic curvature, in: Hermitian–Grassmannian Submanifolds, Springer, Singapore, 2017, pp. 155–164).

RÉSUMÉ

Dans cette note, nous démontrons qu'il n'existe pas d'hypersurface réelle conformément plate dans les espaces de formes complexes de dimension deux, non plats, pourvu que le champ de vecteurs structurel soit champ de vecteur propre de l'opérateur de Ricci. Ceci étend des résultats récents de Cho (Conformally flat normal almost contact 3-manifolds, Honam Math. J. 38 (2016) 59–69) et Kron (3-dimensional real hypersurfaces with η-harmonic curvature, in : Hermitian–Grassmannian Submanifolds, Springer, Singapore, 2017, pp. 155–164).

1. Introduction

A complex n-dimensional Kählerian manifold with constant holomorphic sectional curvature c is said to be a *complex space form* and is denoted by $M^n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ according to $c>0$, $c=0$ or $c<0$, respectively. Let M be a real hypersurface in a complex space form $M^n(c)$, $c\neq 0$, whose Kähler metric and complex structure are denoted by $\overline{g}$ and J, respectively. On M there exists an *almost contact metric structure* (ϕ,ξ,η,g) induced from $\overline{g}$ and J (see Section 2), where ξ is called a *structure vector field*. Let D be the distribution determined by

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https://doi.org/10.1016/j.crma.2018.04.025
1631-073X/

tangent vectors orthogonal to ξ at each point of M. Let A be the shape operator of M in $M^n(c)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$, then M is called a *Hopf hypersurface* and α is called *Hopf principal curvature*.

Let us recall some known results regarding the Weyl conformal tensor on real hypersurfaces. The Riemannian curvature tensor R is harmonic (i.e. $\mathrm{div}R = 0$) if and only if the associated Ricci operator Q is of Codazzi type, i.e.

$$(\nabla_X Q)Y = (\nabla_Y Q)X$$

for any vector fields X, Y. The parallelism of the Ricci tensor implies naturally the harmonicity, but the converse is not necessarily true (see [5]).

Theorem 1.1 *([10,15]). There are no real hypersurfaces with harmonic curvature tensor in a nonflat complex space form $M^n(c)$, $n \geq 2$, on which ξ is principal.*

Theorem 1.1 extends Kimura [11, Theorem 2], who says that there are no real hypersurfaces in $\mathbb{C}P^n(c)$ with parallel Ricci tensor on which ξ is principal. Such conclusion is still true even when the condition "ξ is principal" is removed and the ambient space is generalized to any nonflat space from (see [6, Theorem A]). The curvature tensor is said to be η-harmonic if it satisfies $g((\nabla_X Q)Y - (\nabla_Y Q)X, Z) = 0$ for any vector fields X, Y and Z in D (see [7]). In fact, the η-harmonicity of the curvature tensor on a real hypersurface in complex planes implies η-parallelism of the Ricci tensor under some other restrictions (see Kon [14, Theorem 1]). We remark that the conclusion of Theorem 1.1 is still true if the condition "ξ is principal" is weakened to "ξ is an eigenvector field of the Ricci operator" for dimension three.

Theorem 1.2 *([14]). There are no real hypersurfaces with harmonic curvature tensor in a nonflat complex space form $M^2(c)$ of complex dimension two on which the Ricci operator Q satisfies $Q\xi = \beta\xi$, where β is a function.*

The Weyl conformal tensor W on a Riemannian manifold of dimension greater than three is harmonic (i.e. $\mathrm{div}W = 0$) if the associated Ricci operator satisfies $(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{2(n-1)}(X(r)Y - Y(r)X)$ for any vector fields X, Y, where r denotes the scalar curvature. Therefore, the harmonicity of the Riemannian curvature tensor can be viewed as a special case of that of the Weyl tensor. Such two notions are the same, under condition that the scalar curvature is a constant. Notice that there are Riemannian manifolds on which the Weyl tensor is harmonic but the curvature tensor is not harmonic (see [1]). We observe that Theorem 1.1 was generalized to the following one for dimensions greater than three.

Theorem 1.3 *([9]). There are no real hypersurfaces with harmonic Weyl tensor in a nonflat complex space form $M^n(c)$, $n \geq 3$.*

Generalizing Theorem 1.1, Ki, Kim and Nakagawa in [7] considered a weaker condition named η-harmonicity of the Weyl conformal tensor (i.e. $g((\nabla_X Q)Y - (\nabla_Y Q)X, Z) = \frac{1}{2(n-1)}g(X(r)Y - Y(r)X, Z)$ for any vector fields X, Y, Z orthogonal to the structure vector field ξ, where n is the dimension of the manifold). The authors in [7] also classified real hypersurfaces in a nonflat complex space form $M^n(c)$, $n \geq 3$, provided that ξ is principal and the Weyl tensor is η-harmonic.

Note that the Weyl tensor vanishes on a 3-dimensional Riemannian manifold M^3. Therefore, one always consider another conformal invariant, which is named the Cotton tensor and defined by

$$C(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{X(r)Y - Y(r)X\} \tag{1.1}$$

for any vector fields X, Y on M^3. A Riemannian 3-manifold is conformally flat if and only if the Cotton tensor C vanishes identically. From Theorem 1.3, we know there are no conformally flat real hypersurfaces in a nonflat complex space form $M^n(c)$, $n \geq 3$. Except for the above result, conformally flat hypersurfaces of dimension greater than three in a conformally flat Riemannian manifold were investigated in [19]. However, as far as we know, the studies on conformal flatness on a three-dimensional real hypersurface in complex planes are few. In this paper, we study this problem and prove the following.

Theorem 1.4. *There are no conformally flat real hypersurfaces in nonflat complex space forms of complex dimension two provided that the structure vector field is an eigenvector field of the Ricci operator.*

The condition "ξ is an eigenvector field of the Ricci operator" is rather weak. Such condition was also studied by many authors in recent papers (for example, see [8], [12–14] and [16] and references therein).

Our main result extends naturally Theorems 1.1 and 1.2 in [10,14,15] and is a nice complement of Theorem 1.3 in [9] for dimension three.