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Algebraic geometry

Connections and restrictions to curves

Connexions et restrictions aux courbes

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ABSTRACT

We construct a vector bundle E on a smooth complex projective surface X with the property that the restriction of E to any smooth closed curve in X admits an algebraic connection while E does not admit any algebraic connection.

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RÉSUMÉ

Nous construisons un fibré vectoriel E sur une surface complexe lisse X tel que la restriction de E à toute courbe lisse fermée contenue dans X admet une connexion algébrique, sans que E lui-même admette une telle connexion algébrique.

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1. Introduction

Let X be an irreducible smooth complex projective variety with cotangent bundle Ω^1_X and E a vector bundle on X. The coherent sheaf of local sections of E will also be denoted by E. A connection on E is a k-linear homomorphism of sheaves $D : E \longrightarrow E \otimes \Omega^1_X$ satisfying the Leibniz identity, which says that $D(fs) = fD(s) + s \otimes \mathrm{d}f$, where s is a local section of E and f is a locally defined regular function.

Consider the sheaf of differential operators $\mathrm{Diff}^i_X(E, E)$, of order i on E, and the associated symbol homomorphism $\sigma : \mathrm{Diff}^1_X(E, E) \longrightarrow \mathrm{End}(E) \otimes TX$. The inverse image

$$\mathrm{At}(E) := \sigma^{-1}(\mathrm{Id}_E \otimes TX)$$

is the Atiyah bundle for E. The resulting short exact sequence

$$0 \longrightarrow \mathrm{Diff}^0_X(E, E) = \mathrm{End}(E) \longrightarrow \mathrm{At}(E) \xrightarrow{\sigma} TX \longrightarrow 0 \tag{1.1}$$

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is called the Atiyah exact sequence for E. A connection on E is a splitting of (1.1). We refer the reader to [1] for the details; in particular, see [1, p. 187, Theorem 1] and [1, p. 194, Proposition 9].

When X is a complex curve, Weil and Atiyah proved the following [13], [1]:

A vector bundle V on an irreducible smooth projective curve defined over $\mathbb{C}$ admits a connection if and only if the degree of each indecomposable component of V is zero.

This was first proved in [13]; see also [6, p. 69, THÉORÈME DE WEIL] for an exposition of it. The above criterion also follows from [1, p. 188, Theorem 2], [1, p. 201, Theorem 8] and [1, Theorem 10].

A semistable vector bundle V on a smooth complex projective variety X admits a connection if all the rational Chern classes of E vanish [12, p. 40, Corollary 3.10]. On the other hand, a vector bundle W on X is semistable if and only if the restriction of W to a general complete intersection curve, which is an intersection of hyperplanes of sufficiently large degrees, is semistable [5, p. 637, Theorem 1.2], [11, p. 221, Theorem 6.1]. On the other hand, any vector bundle E whose restriction to every curve is semistable actually satisfies very strong conditions [3]; for example, if X is simply connected, then E must be of the form $L^{\oplus r}$ for some line bundle L.

The following is a natural question to ask.

Question 1.1. Let E be a vector bundle on X such that, for every smooth closed curve $C \subset X$, the restriction $E|_C$ admits a connection. Does E admit a connection?

Our aim is to show that, in general, the above vector bundle E does not admit a connection.

To produce an example of such a vector bundle, we construct a smooth complex projective surface X with $\mathrm{Pic}(X) = \mathbb{Z}$ such that X admits an ample line bundle L_0 with $H^1(X, L_0) \neq 0$. Since $\mathrm{Pic}(X) = \mathbb{Z}$, the ample line bundles on X are naturally parametrized by positive integers. Let L be the smallest ample line bundle (with respect to this parametrization) with the property that $H^1(X, L) \neq 0$. Let E be a nontrivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

We prove that the vector bundle $\mathrm{End}(E)$ has the property that the restriction of it to every smooth closed curve in X admits a connection, while $\mathrm{End}(E)$ does not admit a connection; see Theorem 3.1.

A surface X of the above type is constructed by taking a hyper-Kähler 4-fold X' with $\mathrm{Pic}(X') = \mathbb{Z}$. Let $Y \subset X'$ be a smooth ample hypersurface such that $H^j(X', \mathcal{O}_{X'}(Y)) = 0$ for $j = 1, 2$, and let Z be a very general ample hypersurface of X' such that $H^j(X', \mathcal{O}_{X'}(Z)) = 0$ for $j = 1, 2$ and $H^2(X', \mathcal{O}_{X'}(Z - Y)) = 0$. Now take the surface X to be the intersection $Y \cap Z$.

2. Construction of a surface

We will construct a smooth complex projective surface S with Picard group $\mathbb{Z}$ that has an ample line bundle L with $H^1(S, L) \neq 0$.

Let X be a hyper-Kähler 4-fold with Picard group $\mathbb{Z}$. For example, a sufficiently general deformation of $\mathrm{Hilb}^2(M)$, where M is a polarized $K3$ surface, will have this property. Let $Y \subset X$ be a smooth ample hypersurface. Note that the vanishing theorem of Kodaira says that

$$H^j(X, \mathcal{O}_X(Y)) = 0 \tag{2.1}$$

for all $j > 0$, because K_X is trivial [10]. Let Z be a very general ample hypersurface of X such that both the line bundles $\mathcal{O}_X(Z)$ and $\mathcal{O}_X(Z - Y)$ are ample. In view of the vanishing theorem of Kodaira, the ampleness of $\mathcal{O}_X(Z)$ implies that

$$H^j(X, \mathcal{O}_X(Z)) = 0 \tag{2.2}$$

for all $j > 0$, while that of $\mathcal{O}_X(Z - Y)$ implies that

$$H^j(X, \mathcal{O}_X(Z - Y)) = 0 \tag{2.3}$$

for all $j > 0$. Let

$$\iota : S := Y \cap Z \hookrightarrow X$$

be the intersection and

$$L := \mathcal{O}_X(Y)|_S$$

the restriction of it. Note that L is ample.

Let $\mathcal{I} := \mathcal{O}_X(-S) \subset \mathcal{O}_X$ be the ideal sheaf for S. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_S \longrightarrow 0$$

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