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Spectral localization for quantum Hamiltonians with weak random delta interaction

Localisation spectrale pour des hamiltoniens quantiques, avec une faible interaction aléatoire delta

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ABSTRACT

We consider a negative Laplacian in multi-dimensional Euclidean space (or a multi-dimensional layer) with a weak disorder random perturbation. The perturbation consists of a sum of lattice translates of a delta interaction supported on a compact manifold of co-dimension one and modulated by coupling constants, which are independent identically distributed random variables times a small disorder parameter. We establish that the spectrum of the considered operator is almost surely a fixed set, characterize its minimum, give an initial length scale estimate and the Wegner estimate, and conclude that there is a small zone of a pure point spectrum containing the almost sure spectral bottom. The length of this zone is proportional to the small disorder parameter.

RÉSUMÉ

Nous considérons le laplacien dans un espace euclidien multi-dimensionel (ou dans une couche multi-dimensionelle), avec une perturbation aléatoire à faible désordre. La perturbation consiste en une somme de translations par des points d'un réseau d'une interaction delta, supportée sur une variété de codimension un, qui sont modulées par des variables aléatoires indépendantes et identiquement distribuées, multipliées par un paramètre petit global. Nous démontrons que le spectre de cet opérateur est presque sûrement un ensemble déterministe, nous identifions son minimum spectral, nous donnons une estimation de la longueur de pas initial et une estimée de Wegner, et nous en déduisons qu'il y a une petite zone, contenant le minimum du spectre, dans laquelle ce

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dernier est purement ponctuel. Le diamètre de cette zone est proportionnel au paramètre contrôlant le désordre faible.

1. Introduction

One of the most prominent aspects in the theory of random Hamiltonians is Anderson's localization. From the physical point of view, localization in a disordered medium describes the situation when waves within a certain energy or frequency zone do not propagate through the medium. Mathematically, this means, in particular, that the spectrum of the considered random operator contains a region where almost surely a continuous spectrum is absent and consequently the entire spectrum consists of the closure of the set of eigenvalues.

In this paper, we give the key results and ingredients for Anderson's localization in a new model, namely, random delta-interactions supported on surfaces or, more precisely, on manifolds of co-dimension one. Previously, Hamiltonians with random delta-interactions have been studied only in the case where the support of the singular interactions is discrete, leading to Hamiltonians with point interactions (in low dimensions). To the best of our knowledge, the only two results valid in dimensions larger than one are [8], where Hislop, Kirsch, and Krishna establish localization for random point interactions in dimension up to 3, and [9] where Klopp and Pankrashkin study quantum graphs with random vertex couplings.

The study of localization for Hamiltonians with random singular interactions poses several challenges, which are not present for simple models like the Anderson or alloy-type one. As the very name says, the perturbation is no longer a potential but in fact, a singular measure. Furthermore, at least in the generality which we are treating, it is not required that the perturbation be monotone with respect to the coupling constants. It is well known that non-monotonicity in the random variables requires adapted methods in the context of localization. One possibility to 'tame' the singular interaction is to transform the operator into another one which is spectrally equivalent, but easier to analyze. This is, in fact, the first step of our proof; however, one has to pay a price for this. The transformed operator does no longer depend in a linear fashion on the coupling constants, but in a non-linear one.

The model that we present here is just one instance of a general class of models which we are able to treat and which we will discuss in detail in a subsequent paper [6]. The first steps to implement the multiscale analysis proof of localization for a general random operator was initiated in the works [4], [2]. In these papers, the initial length scale estimate at the bottom of the spectrum was proved for the Schrödinger operator in a multi-dimensional layer with a small random perturbation distributed in the cells of some periodic lattice. The perturbation was described by an abstract operator of the form $\mathcal{L}(t) = t\mathcal{L}_1 + t^2\mathcal{L}_2 + t^3\mathcal{L}_3(t)$ depending on a real parameter t with the coefficients $\mathcal{L}_i$ satisfying certain rather weak conditions. This operator was considered at each periodicity cell, and the parameter t was replaced then by $\varepsilon\omega_k$, where ε is a global small parameter and ω_k is a random variable associated with the cell. By choosing various particular cases of the operators $\mathcal{L}_i$, the proposed model covered many interesting particular examples, both known and new.

A complementary ingredient to complete the multiscale analysis and conclude localization is a Wegner estimate. Since we are working in the weak-disorder regime, both the initial length scale and the Wegner estimate we aim for are restricted to certain zones is the energy $\times$ disorder diagram. For this reason, it is necessary to resolve a preliminary question: is there any spectrum in the zone where our Wegner estimate applies? Of course, it is trivial to prove a Wegner estimate for energies in the resolvent set, but this cannot lead to a localization result in the proper sense of the word.

So the question is whether the spectrum expands under the influence of the random perturbation or not. And if yes: at which rate? This problem was successfully solved in [5] for the above-described general model. This enables us to turn to the final missing step, namely, the Wegner estimate. By employing and modifying the technique proposed in [7], we succeeded in reproving the Wegner estimate. Depending on the rate of the expansion of the spectrum in the weak-disorder regime, the Wegner estimate takes on different forms. The results for the general model will be presented and proved in [6].

As an excerpt and illustration of the aforementioned general results, we treat in the paper a new interesting example, namely, a weak random delta interaction in multi-dimensional Euclidean space (or a layer) where the delta-interaction is supported on a manifold of codimension one. We present an initial length scale and a Wegner estimate for this model and, as a corollary, Anderson's localization at the bottom of the spectrum.

2. Problem and results

Our study applies both to operators defined on the whole Euclidean space as well as to ones on strips, layers, or higher-dimensional analogs. In fact, the results for operators on the whole Euclidean space $\mathbb{R}^n$ can be recovered as a corollary from the ones for operators on layer domains in $\mathbb{R}^{n+1}$, using the reduction described in [4, §3.7]. For this reason, we discuss in the present note strip domains only. Let $x' = (x_1, \ldots, x_n)$, $x = (x', x_{n+1})$, $n \geqslant 1$, be Cartesian coordinates in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively, Π be the multidimensional layer $\Pi := \{x : 0 < x_{n+1} < d\}$ of width $d > 0$. In $\mathbb{R}^n$ we introduce a periodic lattice