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Algebra

Essential dimension of finite groups in prime characteristic [☆]

Dimension essentielle des groupes finis en caractéristique positive

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A R T I C L E I N F O

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A B S T R A C T

Let F be a field of characteristic $p > 0$ and G be a smooth finite algebraic group over F. We compute the essential dimension $\mathrm{ed}_F(G; p)$ of G at p. That is, we show that

$$\mathrm{ed}_F(G;p) = \begin{cases} 1, & \text{if } p \text{ divides } |G|, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

R É S U M É

Soit F un corps de caractéristique $p > 0$, et soit G un groupe algébrique fini étale sur F. On calcule la dimension essentielle de G en p, que l'on note $\mathrm{ed}_F(G; p)$. Plus précisément, on démontre que

$$\mathrm{ed}_F(G;p) = \begin{cases} 1, & \text{si } p \text{ divise } |G|, \\ 0, & \text{sinon.} \end{cases}$$

1. Introduction

Let F be a field and G be an algebraic group over F. We begin by recalling the definition of the essential dimension of G.

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Let K be a field containing F and $\tau : T \to \mathrm{Spec}(K)$ be a G-torsor. We will say that τ descends to an intermediate subfield $F \subset K_0 \subset K$ if τ is the pull-back of some G-torsor $\tau_0 : T_0 \to \mathrm{Spec}(K_0)$, i.e. if there exists a Cartesian diagram of the form

$$\begin{array}{ccccc} T & \longrightarrow & T_0 & & \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau_0} & & \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(K_0) & \longrightarrow & \mathrm{Spec}(F). \end{array}$$

The essential dimension of τ, denoted by $\mathrm{ed}_F(\tau)$, is the smallest value of the transcendence degree $\mathrm{trdeg}(K_0/F)$ such that τ descends to K_0. The essential dimension of G, denoted by $\mathrm{ed}_F(G)$, is the maximal value of $\mathrm{ed}_F(\tau)$, as K ranges over all fields containing F and τ ranges over all G-torsors $T \to \mathrm{Spec}(K)$.

Now let p be a prime integer. A field K is called p-closed if the degree of every finite extension L/K is a power of p. Equivalently, $\mathrm{Gal}(K^s/K)$ is a pro-p-group, where K^s is a separable closure of K. For example, the field of real numbers is 2-closed. The essential dimension $\mathrm{ed}_F(G; p)$ of G at p is the maximal value of $\mathrm{ed}_F(\tau)$, where K ranges over p-closed fields K containing F, and τ ranges over the G-torsors $T \to \mathrm{Spec}(K)$. For an overview of the theory of essential dimension, we refer the reader to the surveys [19] and [16].

The case where G is a finite group (viewed as a constant group over F) is of particular interest. A theorem of N.A. Karpenko and A.S. Merkurjev [10] asserts that, in this case,

$$\mathrm{ed}_F(G; p) = \mathrm{ed}_F(G_p; p) = \mathrm{ed}_F(G_p) = \mathrm{rdim}_F(G_p)\,, \qquad (1)$$

provided that F contains a primitive p-th root of unity ζ_p. Here G_p is any Sylow p-subgroup of G, and $\mathrm{rdim}_F(G_p)$ denotes the minimal dimension of a faithful representation of G_p defined over F. For example, assuming that $\zeta_p \in F$, $\mathrm{ed}_F(G) = \mathrm{ed}(G; p) = r$ if $G = (\mathbb{Z}/p\mathbb{Z})^r$, and $\mathrm{ed}(G) = \mathrm{ed}(G; p) = p$ if G is a non-abelian group of order p^3. Further examples can be found in [18].

Little is known about essential dimension of finite groups over a field F of characteristic $p > 0$. A. Ledet [12] conjectured that

$$\mathrm{ed}_F(\mathbb{Z}/p^r\mathbb{Z}) = r \qquad (2)$$

for every $r \geqslant 1$. This conjecture remains open for every $r \geqslant 3$. In this paper we will prove the following surprising result.

Theorem 1. *Let F be a field of characteristic $p > 0$ and G be a smooth finite algebraic group over F. Then*

$$\mathrm{ed}_F(G; p) = \begin{cases} 1, & \text{if } p \text{ divides } |G|, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, Ledet's conjecture (2) fails dramatically if the essential dimension is replaced by the essential dimension at p. On the other hand, Theorem 1 fails if $\mathrm{ed}(G; p)$ is replaced by $\mathrm{ed}(G)$; see [13].

Before proceeding with the proof of Theorem 1, we remark that the condition that G is smooth cannot be dropped. Indeed, it is well known that $\mathrm{ed}_F(\mu_p^r; p) = r$ for any $r \geqslant 0$. More generally, if G is a group scheme of finite type over a field F of characteristic p (not necessarily finite or smooth), then $\mathrm{ed}_F(G; p) \geqslant \dim(\mathcal{G}) - \dim(G)$, where $\mathcal{G}$ is the Lie algebra of G; see [25, Theorem 1.2].

2. Versality

Let G be an algebraic group and X be an irreducible G-variety (i.e. a variety with a G-action) over F. We will say that the G-variety X is *generically free* if there exists a dense open subvariety U of X such that the scheme-theoretic stabilizer G_u of every geometric point u of X is trivial. Equivalently, there exists a G-invariant dense open subvariety U' of X, which is the total space of a G-torsor; see [23, Section 5].

Following [23, Section 5] and [6, Section 1], we will say that X is *weakly versal* (respectively, *weakly p-versal*) if, for every infinite field (respectively, every p-closed field) E, and every G-torsor $T \to \mathrm{Spec}(E)$, there is a G-equivariant F-morphism $T \to X$. We will say that X is *versal* (respectively, p-versal), if every G-invariant dense open subvariety of X is weakly versal (respectively, weakly p-versal).

It readily follows from these definitions that $\mathrm{ed}(G)$ (respectively, $\mathrm{ed}(G; p)$) is the minimal dimension $\dim(X) - \dim(G)$, where the minimum is taken over all versal (respectively p-versal) generically free G-varieties X; see [23, Section 5.7], [6, Remark 2.6 and Section 8]. Our proof of Theorem 1 will be based on the following facts.

(i) ([6, Proposition 2.2]) Every G-variety X with a G-fixed F-point is weakly versal.

(ii) ([6, Theorem 8.3]) Let X be a smooth geometrically irreducible G-variety. Then X is weakly p-versal if and only if X is p-versal.