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Tiltan

Trèfle

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ABSTRACT

We prove the consistency of $\clubsuit$ with the negation of Galvin's property. On the other hand, we show that superclub implies Galvin's property. We also prove the consistency of $\clubsuit_{\kappa^+}$ with $\mathfrak{s}_\kappa > \kappa^+$ for a supercompact cardinal κ.

RÉSUMÉ

Nous démontrons que le principe trèfle $\clubsuit$ et la négation de la propriété de Galvin sont consistants. D'un autre côté, nous montrons que supertrèfle implique la propriété de Galvin. Nous montrons également que $\clubsuit_{\kappa^+}$ et $\mathfrak{s}_\kappa > \kappa^+$ sont consistants pour un cardinal supercompact κ.

1. Introduction

The diamond principle of Jensen [9] is a prediction principle. It says that there exists a sequence of sets $(A_\alpha : \alpha \in \omega_1)$ such that each A_α is a subset of α, and such that, for every $A \subseteq \omega_1$, the set $\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\}$ is a stationary subset of ω_1.

A weaker prediction principle, denoted by $\clubsuit$, was introduced by Ostaszewski in [11]. Usually it is called the club principle, but we shall employ the name *tiltan* to refer to $\clubsuit$. The reason is that we are going to deal extensively with closed unbounded sets using the acronym *club*, and anticipating a natural confusion we prefer a linguistic distinction. The name tiltan means clover in Mishnaic Hebrew. The local version of tiltan at $\aleph_1$ reads as follows.

Definition 1.1 *(Tiltan).* There exists a sequence of sets $\langle T_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that each T_α is a cofinal subset of α, and such that, for every unbounded set $A \subseteq \omega_1$, there are stationarily many ordinals α for which $T_\alpha \subseteq A$.

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The definition generalizes easily to any stationary set S of any regular uncountable cardinal κ whose elements are limit ordinals. The tiltan sequence will be $\langle T_\alpha : \alpha \in S\rangle$, and the assertion will be denoted by $\clubsuit_S$. Clearly, if $\clubsuit_{S_0}$ holds and $S_1 \supseteq S_0$, then $\clubsuit_{S_1}$ holds as well.

It is clear from the definition that $\diamondsuit \Rightarrow \clubsuit$. The difference is two-fold. Firstly, the $\diamondsuit$-prediction is accurate and based on the equality relation, namely $A \cap \alpha = A_\alpha$, while the $\clubsuit$-prediction promises only inclusion, i.e. $T_\alpha \subseteq A$. Secondly, the diamond predicts *all* the subsets of ω_1 (or larger cardinals) including the countable subsets, while the tiltan predicts only *unbounded* subsets of ω_1. In both points, $\clubsuit$ is weaker than $\diamondsuit$.

One may wonder if the tiltan is strictly weaker than the diamond. It is easy to show that $\clubsuit + 2^\omega = \omega_1$ is equivalent to $\diamondsuit$. The question reduces, therefore, to the possible consistency of tiltan with $2^\omega > \omega_1$. The answer is yes, as proved by Shelah in [14], I, §7. The proof shows, in particular, the consistency of $\clubsuit + \neg\diamondsuit$. This result opens a window to a variety of consistency results of this form.

Suppose that φ is a mathematical statement that follows from the diamond. One may ask whether the tiltan is consistent with $\neg\varphi$. We focus, in this paper, on a statement which we call *Galvin's property*. It is based on a theorem of Galvin that appears in [2]. We quote the version of $\aleph_1$ and club sets, but the theorem generalizes to every normal filter over any regular uncountable cardinal.

Theorem 1.2 *(CH and Galvin's property). Assume that* $2^\omega = \omega_1$.

Then any collection $\{C_\alpha : \alpha \in \omega_2\}$ *of club subsets of* $\aleph_1$ *admits a sub-collection* $\{C_{\alpha_\beta} : \beta \in \omega_1\}$ *whose intersection is a club subset of* $\aleph_1$. $\square_{1.2}$

Galvin's property follows from CH, and a fortiori from the diamond. Question 2.4 from [5] is whether Galvin's property follows from the tiltan. The original proof of Galvin gives the impression that the answer should be positive. Surprisingly, we shall prove the opposite by showing the consistency of tiltan with the failure of Galvin's property.

Nevertheless, something from the natural impression still remains and can be proved. Tiltan is consistent only with a weak negation of Galvin's property. The strong negation of it cannot be true under the tiltan assumption. Let us try to clarify this point.

Galvin's property deals with a sub-collection whose intersection is a club, but the real point is only unboundedness. If $C = \bigcap\{C_{\alpha_\beta} : \beta \in \omega_1\}$ and $a \subseteq C$ is unbounded, then $c\ell(a) \subseteq C$ as well. Consequently, if one wishes to force the negation of Galvin's property, then a bounded intersection must be forced. This is done, twice, in a work of Abraham and Shelah [1]. Our purpose is to combine the forcing of [1] with the classical way to force $\clubsuit + \neg$CH, thus obtaining the main result of the next section:

Theorem 1.3. *It is consistent that* $\clubsuit$ *holds,* $2^\omega = \lambda$, λ *is arbitrarily large, and there exists a collection* $\{C_\alpha : \alpha < \lambda\}$ *of club subsets of* $\aleph_1$ *such that any* $\aleph_1$*-sub-collection of it has bounded intersection.* $\square_{1.3}$

The negation of Galvin's property, reflected in the above theorem, is different from the situation in [1] notwithstanding. In the constructions of [1] not only any sub-collection of size $\aleph_1$ has bounded intersection (in $\aleph_1$), but it has *finite* intersection. Let us call this property *a strong negation of Galvin's property*. We shall see that tiltan is incompatible with such a strong negation. Namely, under $\clubsuit$ any collection of the form $\{C_\alpha : \alpha < \omega_2\}$ contains even a sub-collection of $\aleph_2$-many sets with infinite intersection. Actually, an intersection of order type $\geq \tau$ for every ordinal $\tau \in \omega \cdot \omega$ can be shown to exist. This means that the main theorem is optimal in some sense. Moreover, it gives some information about possible ways to force tiltan and their limitations. One way to demonstrate this observation is to strengthen tiltan, as done in the second section. We shall work with the prediction principle *superclub* from [12] and show that it implies Galvin's property. In the last section, we deal with the splitting number $\mathfrak{s}_\kappa$ and the possibility that $\clubsuit_{\kappa^+}$ be consistent with $\mathfrak{s}_\kappa > \kappa^+$.

Our notation is mostly standard. If $\kappa = \mathrm{cf}(\kappa) < \lambda$, then $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. If $\mathrm{cf}(\lambda) > \omega$ then S^λ_κ is a stationary subset of λ. We shall use the Jerusalem forcing notation, namely $p \leq q$ means that p is weaker than q. If $\mathcal{I}$ is an ideal over κ then $\mathcal{I}^+ = \mathcal{P}(\kappa) - \mathcal{I}$. We shall always assume that every bounded subset of κ belongs to $\mathcal{I}$. The notation NS_κ refers to the non-stationary ideal over κ.

Suppose that $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $S \subseteq \kappa$, S is stationary and $\diamondsuit_S$ holds, as exemplified by $\langle A_\delta : \delta \in S\rangle$. Suppose that M is any structure over κ, and the size of $\mathcal{L}(M)$ is at most κ. We would like to say that the diamond sequence predicts elementary submodels of M. However, the diamond sequence predicts sets of ordinals, and M contains many objects that are not ordinals.

It is possible to code all the information in M as subsets of κ. For this, we fix $|\mathcal{L}(M)|$ disjoint subsets of κ, each of which is of size κ, denoted by $\{B_R : R \in \mathcal{L}(M)\}$. We also fix one-to-one functions from $\kappa^{n(R)}$ into B_R for every $R \in \mathcal{L}(M)$ where $n(R) = \mathrm{arity}(R)$. The union of the range of these functions is a subset of κ, hence the sequence $\langle A_\delta : \delta \in S\rangle$ predicts it at stationarily many places.

Since the code of each R^M lies in a set disjoint from the other sets and the functions are one-to-one, it is possible to decode the information and recover a submodel of M at each point in which the diamond sequence guesses an initial segment of the above set. Moreover, the set of ordinals for which such a submodel is elementary will be still a stationary set. We indicate that the same diamond sequence predicts, in this way, elementary submodels of every structure over κ.