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Partial differential equations/Mathematical physics

On maximizing the fundamental frequency of the complement of an obstacle

Sur la maximisation de la fréquence fondamentale du complément d'un obstacle

Bogdan Georgiev [a], Mayukh Mukherjee [b]

[a] *Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany*
[b] *Mathematics Department, Technion – I.I.T., Haifa 32000, Israel*

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying a Hayman-type asymmetry condition, and let D be an arbitrary bounded domain referred to as an "obstacle". We are interested in the behavior of the first Dirichlet eigenvalue $\lambda_1(\Omega \setminus (x+D))$.
First, we prove an upper bound on $\lambda_1(\Omega \setminus (x+D))$ in terms of the distance of the set $x+D$ to the set of maximum points x_0 of the first Dirichlet ground state $\phi_{\lambda_1} > 0$ of Ω. In short, a direct corollary is that if

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x+D)) \tag{1}$$

is large enough in terms of $\lambda_1(\Omega)$, then all maximizer sets $x+D$ of μ_Ω are close to each maximum point x_0 of ϕ_{λ_1}.
Second, we discuss the distribution of $\phi_{\lambda_1(\Omega)}$ and the possibility to inscribe wavelength balls at a given point in Ω.
Finally, we specify our observations to convex obstacles D and show that if μ_Ω is sufficiently large with respect to $\lambda_1(\Omega)$, then all maximizers $x+D$ of μ_Ω contain all maximum points x_0 of $\phi_{\lambda_1(\Omega)}$.

RÉSUMÉ

Soit $\Omega \subset \mathbf{R}^n$ un domaine borné satisfaisant une condition de type Hayman asymétrique et soit D un domaine borné arbitraire, dénommé « obstacle ». Nous nous intéressons au comportement de la première valeur propre de Dirichlet $\lambda_1(\Omega \setminus (x+D))$.
Nous établissons, dans un premier temps, une borne supérieure pour cette valeur propre en termes de la distance de l'ensemble $x+D$ à l'ensemble des points x_0 où la fonction propre du premier état de base de Dirichlet $\phi_{\lambda_1} > 0$ de Ω atteint son maximum. En bref, un corollaire immédiat est que, si

$$\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x+D))$$

E-mail addresses: bogeor@mpim-bonn.mpg.de (B. Georgiev), mathmukherjee@gmail.com (M. Mukherjee).

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est suffisamment grand en fonction de $\lambda_1(\Omega)$, alors tous les ensembles maximisant $x+D$ de μ_Ω sont proches de chaque point x_0 où ϕ_{λ_1} est maximum.
Ensuite, nous discutons la distribution de $\phi_{\lambda_1(\Omega)}$ et la possibilité d'inscrire des boules de longueur d'onde en un point donné de Ω.
Enfin, nous appliquons nos observations aux obstacles convexes D, et nous montrons que, si μ_Ω est suffisamment grand par rapport à $\lambda_1(\Omega)$, alors tous les ensembles maximisant $x+D$ de μ_Ω contiennent tous les points x_0 où $\phi_{\lambda_1(\Omega)}$ est maximum.

1. Introduction and background

We consider the natural problem (seemingly first posed by Davies) of placing an obstacle in a domain so as to maximize the fundamental frequency of the complement of the obstacle. To be more precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let D be another bounded domain referred to as an "obstacle". The problem is to determine the optimal translate $x+D$ so that the fundamental Dirichlet Laplacian eigenvalue $\lambda_1(\Omega \setminus (x+D))$ is maximized/minimized.

In case the obstacle D is a ball, physical intuition suggests that for sufficiently regular domains and sufficiently small balls, Ω, $\lambda_1(\Omega \setminus B_r(x))$ will be maximized when $x = x_0$, a point of maximum of the ground state Dirichlet eigenfunction ϕ_{λ_1} of Ω. Heuristically, such maximum points x_0 seem to be situated deeply in Ω, hence removing a ball around x_0 should be an optimal way of truncating the lowest possible frequency. Our methods give equally good results for Schrödinger operators on a large class of bounded domains sitting inside Riemannian manifolds (see the remarks at the end of Section 2).

The following well-known result of Harrell–Kröger–Kurata treats the case when Ω satisfies convexity and symmetry conditions.

Theorem 1.1 *([11]). Let Ω be a convex domain in $\mathbb{R}^n$ and B a ball contained in Ω. Assume that Ω is symmetric with respect to some hyperplane H. Then,*

(a) at the maximizing position, B is centered on H, and
(b) at the minimizing position, B touches the boundary of Ω.

The last result of Harrell–Kröger–Kurata seems to work under a rather strong symmetry assumption. We also recall that the proof of Harrell–Kröger–Kurata proceeds via a moving planes method, which essentially measures the derivative of $\lambda_1(\Omega \setminus B)$ when B is shifted in a normal direction to the hyperplane (also see p. 58 of [13]). See also related work in [4], [14].

There does not seem to be any result in the literature treating domains without symmetry or convexity properties.

In our note, we consider bounded domains $\Omega \subset \mathbb{R}^n$ that satisfy an asymmetry assumption in the following sense.

Definition 1.2. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the asymmetry assumption with coefficient α (or Ω is α-asymmetric) if for all $x \in \partial\Omega$, and all $r_0 > 0$,

$$\frac{|B_{r_0}(x) \setminus \Omega|}{|B_{r_0}(x)|} \geq \alpha. \tag{2}$$

This condition seems to have been introduced in [12]. Further, the α-asymmetry property was utilized by D. Mangoubi in order to obtain inradius bounds for Laplacian nodal domains (cf. [16]) as nodal domains are asymmetric with $\alpha = \frac{C}{\lambda^{(n-1)/2}}$.

From our perspective, the notion of asymmetry is useful as it basically rules out narrow "spikes" (i.e. with relatively small volume) entering deeply into Ω. For example, let us also observe that convex domains trivially satisfy our asymmetry assumption with coefficient $\alpha = \frac{1}{2}$.

2. The basic estimate for general obstacles

With the above in mind, we consider any bounded α-asymmetric domain $\Omega \subset \mathbb{R}^n$ and a bounded obstacle domain D. We denote the first positive Dirichlet eigenvalue and eigenfunction of Ω by λ_1 and $\phi_{\lambda_1(\Omega)}$ respectively and let

$$M := \{x \in \Omega \mid \phi_{\lambda_1}(x) = \|\phi_{\lambda_1(\Omega)}\|_{L^\infty(\Omega)}\} \tag{3}$$

be the set of maximum points of $\phi_{\lambda_1(\Omega)}$.