C. R. Acad. Sci. Paris, Ser. I 356 (2018) 235–242

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Combinatorics/Ordinary differential equations

# Majoration of the dimension of the space of concatenated solutions to a specific pantograph equation

## *Majoration de la dimension de l'espace des solutions concaténées d'un cas particulier de l'équation du pantographe*

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A R T I C L E   I N F O

*Article history:*
Received 27 September 2017
Accepted after revision 22 January 2018
Available online 1 February 2018

Presented by the Editorial Board

A B S T R A C T

For each $\lambda \in \mathbb{N}^*$, we consider the integral equation:

$$\int_{\lambda y}^{\lambda x} f(t)\,\mathrm{d}t = f(x) - f(y) \text{ for every } (x, y) \in \mathbb{R}_+^2,$$

where $f$ is the concatenation of two continuous functions $f_a, f_b : [0, \lambda] \to \mathbb{R}$ along a word $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ such that $u = \sigma(u)$, where $\sigma$ is a $\lambda$-uniform substitution satisfying some combinatorial conditions.

There exists some non-trivial solutions ([1]). We show in this work that the dimension of the set of solutions is at most two.



R É S U M É

Nous considérons les équations intégrales de la forme suivante pour $\lambda \in \mathbb{N}^*$ :

$$\int_{\lambda y}^{\lambda x} f(t)\,\mathrm{d}t = f(x) - f(y) \text{ for every } (x, y) \in \mathbb{R}_+^2,$$

où $f$ est la concaténation de deux fonctions continues $f_a, f_b : [0, \lambda] \to \mathbb{R}$ le long d'un mot infini $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ tel que $u = \sigma(u)$, où $\sigma$ est une substitution $\lambda$-uniforme vérifiant certaines propriétés combinatoires.

Il existe des solutions non triviales à ces équations ([1]). Nous montrons dans ce travail que l'espace des solutions est de dimension au plus 2.



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https://doi.org/10.1016/j.crma.2018.01.013
1631-073X/

## 1. Introduction

For each positive integer $\lambda \geq 2$ and each integer $\delta \in \mathbb{Z}^*$, we consider the integral equation:

$$\int_{\lambda y}^{\lambda x} f(t)\,\mathrm{d}t = \delta\big(f(x) - f(y)\big) \text{ for every } (x, y) \in (\mathbb{R}_+)^2. \tag{$E_{\lambda,\delta}$}$$

This equation is a particular case of the *pantograph* equation:

$$f'(x) = af(\tau x) + bf(x) \quad \text{with } (a, b) \in \mathbb{R}^2 \text{ and } \tau \in \mathbb{R}_+ \text{ for } x \geq 0.$$

We refer to [2], [4], [5] and [6] for more details on the pantograph equation.

We prove in [1] that we can extend each continuous function $f$ defined on $[1, \lambda]$ such that $f^{(n)}(1) = f^{(n)}(\lambda) = 0$ for every non-negative integer $n$, into a continuous solution to $(E_{\lambda,\delta})$. Therefore, the set of continuous solutions to $(E_{\lambda,\delta})$ is an infinite-dimensional vector space.

Moreover, we prove in [1] that the non-identically zero solutions are not periodic. It seems natural to look for the simplest solutions to $(E_{\lambda,\delta})$. The periodic functions are the repetition of the same motif. We study the functions that are the repetition (not periodically) of two functions. This leads us to the following notion of *concatenation* of two functions along a word.

**Definition 1.1.** Let $\lambda \geq 2$ be a positive integer and $f_a, f_b : [0, \lambda] \to \mathbb{R}$ be two functions. For each finite word $u = u_0 \cdots u_{n-1} \in \{a, b\}^n$ of length $n$, we define a function $f_u : [0, n\lambda] \to \mathbb{R}$ called *the concatenation of* $f_a$ and $f_b$ along $u$ by:

$$f_u(x + \lambda k) := f_{u_k}(x) \text{ for } x \in [0, \lambda] \text{ and } k \in \{0, \ldots, n-1\}.$$

We extend this definition to infinite words.

Our main result is the following theorem. We recall in Section 2 some notions of combinatorics on words requisite to fully understand this result.

**Theorem 1.2.** *We consider a $\lambda$-uniform substitution $\sigma$, satisfying some combinatorial conditions (Relations (1) and (3)) and $u = u_0 u_1 \cdots \in \{a, b\}^{\mathbb{N}}$ an infinite word such that $u = \sigma(u)$.*

*We consider the integral equation:*

$$\int_{\lambda y}^{\lambda x} f(t)\,\mathrm{d}t = f(x) - f(y) \text{ for every } (x, y) \in (\mathbb{R}_+)^2. \tag{$E_\lambda$}$$

*We denote by $\mathcal{S}_\lambda$ the set of solutions $f$ to $(E_\lambda)$ that are the concatenation of two continuous functions $f_a, f_b : [0, \lambda] \to \mathbb{R}$ along the word $u$. Then $\mathcal{S}_\lambda$ is a vector space of dimension at most 2.*

We prove in [1] that $\mathcal{S}_\lambda$ is of dimension at least 1. To construct a non-trivial solution, we renormalized some iterated Birkhoff sums. The technique used to prove Theorem 1.2 (in Section 5) is very different. It is based on the relation between the values taken by the functions and their moments. This brings us back to the historical first non-trivial solution associated with the Prouhet–Thue–Morse substitution ($a \to ab$ and $b \to ba$) constructed by Fabius ([3]) as a cumulative distribution function.

We do not have examples of substitutions for which the dimension of $\mathcal{S}_\lambda$ is two.

We will use the two following basic results (see [1]).

**Remark 1.** Let $f$ be as in Theorem 1.2, then for every finite word $v$ of length $n$:

$$\int_{\lambda y}^{\lambda x} f_{\sigma(v)}(t)\,\mathrm{d}t = f_v(x) - f_v(y) \text{ for every } (x, y) \in [0, n\lambda]^2.$$

**Remark 2.** We have $f_a(0) = f_a(\lambda) = f_b(0) = f_b(\lambda)$.