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Improved version of Bohr's inequality

Version améliorée de l'inégalité de Bohr

Ilgiz R. Kayumov [a], Saminathan Ponnusamy [b]

[a] *Kazan Federal University, Kremlevskaya 18, 420 008 Kazan, Russia*
[b] *Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India*

A R T I C L E   I N F O

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A B S T R A C T

In this article, we prove several different improved versions of the classical Bohr's inequality. All the results are proved to be sharp.

R É S U M É

Nous montrons ici plusieurs améliorations de l'inégalité de Bohr classique. Nous montrons également que les constantes numériques dans nos résultats sont optimales.

1. Introduction and main results

The classical theorem of Bohr [3] (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if f is a bounded analytic function on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with the Taylor expansion $\sum_{k=0}^{\infty} a_k z^k$, and $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty$, then

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq \|f\|_\infty \quad \text{for } 0 \leq r \leq 1/3 \tag{1}$$

and the constant $1/3$ is sharp. There are a number of articles that deal with Bohr's phenomenon. See, for example, [2,10], the recent survey on this topic by Abu-Muhanna et al. [1] and the references therein. Bombieri [4] considered the function $m(r)$ defined by $m(r) = \sup\left\{M_f(r)/\|f\|_\infty\right\}$, where the supremum is taken over all nonzero bounded analytic functions, and proved that

$$m(r) = \frac{3 - \sqrt{8(1-r^2)}}{r} \quad \text{for } 1/3 \leq r \leq 1/\sqrt{2}.$$

E-mail addresses: ikayumov@kpfu.ru (I.R. Kayumov), samy@iitm.ac.in, samy@isichennai.res.in (S. Ponnusamy).

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Later Bombieri and Bourgain [5] studied the behaviour of $m(r)$ as $r \to 1$ (see also [6]) and proved the following result, which validated a question raised in [11, Remark 1] in the affirmative.

Theorem A. ([5, Theorem 1]) *If $r > 1/\sqrt{2}$, then $m(r) < 1/\sqrt{1-r^2}$. With $\alpha = 1/\sqrt{2}$, the function $\varphi_\alpha(z) = (\alpha - z)/(1 - \alpha z)$ is extremal, giving $m(1/\sqrt{2}) = \sqrt{2}$.*

A lower estimate for $m(r)$ as $r \to 1$ is also obtained in [5, Theorem 2]. We are now ready to state several different improved versions of the classical Bohr inequality (1).

Theorem 1. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$, $|f(z)| \le 1$ in $\mathbb{D}$, and S_r denotes the area of the Riemann surface of the function f^{-1} defined on the image of the subdisk $|z| < r$ under the mapping f. Then*

$$B_1(r) := \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for } r \le \frac{1}{3} \tag{2}$$

and the numbers $1/3$ and $16/9$ cannot be improved. Moreover,

$$B_2(r) := |a_0|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{9}{8}\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for } r \le \frac{1}{2} \tag{3}$$

and the constants $1/2$ and $9/8$ cannot be improved.

Remark 1. Let us remark that if f is a univalent function then S_r is the area of the image of the subdisk $|z| < r$ under the mapping f. In the case of multivalent function, S_r is greater than the area of the image of the subdisk $|z| < r$. This fact could be shown by noting that

$$S_r = \int_{f(\mathbb{D}_r)} |f'(z)|^2 \, dA(w) = \int_{f(\mathbb{D}_r)} \nu_f(w) \, dA(w) \ge \int_{f(\mathbb{D}_r)} dA(w) = \text{Area}(f(\mathbb{D}_r)),$$

where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\nu_f(w) = \sum_{f(z)=w} 1$ denotes the counting function of f.

Theorem 2. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|a_0| + \sum_{k=1}^{\infty} \left(|a_k| + \frac{1}{2}|a_k|^2\right) r^k \le 1 \quad \text{for } r \le \frac{1}{3} \tag{4}$$

and the numbers $1/3$ and $1/2$ cannot be improved.

Theorem 3. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$\sum_{k=0}^{\infty} |a_k| r^k + |f(z) - a_0|^2 \le 1 \quad \text{for } r \le \frac{1}{3}$$

and the number $1/3$ cannot be improved.

Finally, we also prove the following sharp inequality.

Theorem 4. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \le 1 \quad \text{for } r \le \sqrt{\frac{11}{27}} = 0.63828\ldots$$

and this number cannot be improved.