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Differential geometry

On an inequality of Brendle in the hyperbolic space

Sur une inégalité de Brendle dans l'espace hyperbolique

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ABSTRACT

We give a spinorial proof of a Heintze–Karcher-type inequality in the hyperbolic space proved by Brendle [4]. The proof relies on a generalized Reilly formula on spinors recently obtained in [7].

RÉSUMÉ

On donne une nouvelle démonstration d'une inégalité de type Heintze–Karcher dans l'espace hyperbolique prouvée par Brendle [4]. Cette preuve repose sur une formule de Reilly généralisée pour l'opérateur de Dirac, que nous avons récemment obtenue dans [7].

1. Introduction

The classical Alexandrov theorem [1] asserts that, if Σ is a closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature, then Σ is a round sphere. There are different proofs and generalizations of this theorem. Here we are interested in that of [12] (inspired by Reilly's proof [11]), which relies on the following Heintze–Karcher-type inequality

$$n\int_\Sigma \frac{1}{H}\,\mathrm{d}\Sigma \geq -\int_\Sigma \langle \xi, N\rangle\,\mathrm{d}\Sigma \tag{1}$$

which holds for all closed embedded mean convex hypersurfaces Σ in $\mathbb{R}^{n+1}$. Here ξ, N and H denote respectively the position vector field, the unit inner vector field normal to Σ, and the mean curvature of Σ (with our conventions, the unit sphere in $\mathbb{R}^{n+1}$ satisfies $H = n$). Moreover, equality holds if and only if Σ is a round sphere. Now assuming the constancy of the mean curvature H, which has to be positive since Σ is compact, the classical Minkowski formula

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$$n\int_\Sigma \mathrm{d}\Sigma = -\int_\Sigma H\langle \xi, N\rangle\,\mathrm{d}\Sigma$$

implies that equality is achieved in (1) and thus the Alexandrov theorem is proved.

More recently, Brendle [4] generalized this inequality for domains in certain warped product manifolds, which, in particular, include the classical space forms. This formula is then used to prove some unicity theorems for constant mean curvature hypersurfaces in such manifolds generalizing results from the second author [9]. In the hyperbolic space, this inequality reads as:

Theorem 1. *Let $\Omega \subset \mathbb{H}^{n+1}$ be a compact $(n+1)$-dimensional domain with smooth boundary Σ and let $V(x) = \cosh \mathrm{dist}_{\mathbb{H}^{n+1}}(x, b)$ for some fixed point $b \in \Omega$. If Σ is mean convex, then*

$$n\int_\Sigma \frac{V}{H}\,\mathrm{d}\Sigma \geq (n+1)\int_\Omega V\,\mathrm{d}\Omega. \tag{2}$$

The equality in (2) holds if and only if Ω is isometric to a geodesic ball.

The proof of this result relies on the fact that the quantity $\int_\Sigma \frac{V}{H}\,\mathrm{d}\Sigma$ is monotone non-increasing along a geometric flow. It is worth noticing that this approach has recently been successfully adapted in several works. For example, in [13], the authors obtain an analogous formula for codimension-two submanifolds in some warped products spacetime that admit a conformal Killing–Yano 2-form. Then as an application, they prove generalizations of the Alexandrov theorem for adapted curvature conditions in this context. Note that in our work [7], we weaken their assumptions when the spacetime is the Minkowski space. In a same manner, Li and Xia [8] (see also [10]) were able to prove a Heintze–Karcher-type inequality for sub-static manifolds. Their proof relies on a generalization of the classical Reilly formula and fits more with the approach developed by Reilly and Ros.

In this note, we prove that, in the case of the hyperbolic space, inequality (2) is a simple consequence of a generalized Reilly-type inequality on spinors. It follows the spinorial proof of the Heintze–Karcher inequality in $\mathbb{R}^{n+1}$ by Desmonts [5,6].

2. An integral inequality from [7]

Here we specialize an integral inequality proved in our work [7] for certain boundaries of spacelike domains in spacetimes satisfying the Einstein equation and the dominant energy condition in the case of the Minkowski spacetime $\mathbb{R}^{n+1,1}$. For more details, we refer the reader to [7].

Let Ω be a compact $(n+1)$-dimensional domain with smooth boundary Σ of the hyperbolic space $\mathbb{H}^{n+1}$ realized as the spacelike hypersurface of the Minkowski spacetime $\mathbb{R}^{n+1,1}$ defined by

$$\mathbb{H}^{n+1} = \{(x_0, x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+2} \,/\, -x_0^2 + \sum_{i=1}^{n+1} x_i^2 = -1\}.$$

Since $\Omega \subset \mathbb{H}^{n+1}$, the position vector ξ in $\mathbb{R}^{n+1,1}$ is a future-directed timelike vector field normal to Ω in $\mathbb{R}^{n+1,1}$. We will also denote by N the inner unit spacelike vector field normal to Σ in Ω. In this frame, the second fundamental form of Σ^n in $\mathbb{R}^{n+1,1}$ is given by

$$\mathrm{II}(X, Y) = \langle AX, Y\rangle N + \langle X, Y\rangle \xi$$

for all $X, Y \in \Gamma(T\Sigma)$ and where $AX := -\nabla^\Omega_X N$ denotes the shape operator of Σ in Ω. Here ∇^Ω denotes the Levi-Civita connection of the induced Riemannian metric $\langle\ ,\ \rangle$ on Ω. The mean curvature vector field $\mathcal{H}$ of Σ in $\mathbb{R}^{n+1,1}$, defined by $\mathcal{H} = \mathrm{tr}\,\mathrm{II}$, can be expressed as:

$$\mathcal{H} = HN + n\xi$$

where $H = \mathrm{tr}\,A$ is the mean curvature of Σ in Ω.

On $\mathbb{R}^{n+1,1}$, we define the bundle of complex spinors as being the trivial vector bundle $\mathbb{S}\mathbb{R}^{n+1,1} = \mathbb{R}^{n+2} \times \mathbb{C}^m$ where $m = 2^{[\frac{n+2}{2}]}$. The natural action of $\omega \in \mathbb{C}l(\mathbb{R}^{n+1,1})$, an element of the complex Clifford bundle over $\mathbb{R}^{n+1,1}$, on a spinor field $\psi \in \Gamma\big(\mathbb{S}\mathbb{R}^{n+1,1}\big)$ will be denoted by $\widetilde{\gamma}(\omega)\psi$. The existence of a unit timelike vector ξ normal to $\mathbb{H}^{n+1}$ (hence to Ω) allows us to define the restricted spinor bundle over Ω by $\mathbb{S}\!\!\!/\Omega = \mathbb{S}\mathbb{R}^{n+1,1}{}_{|\Omega}$. According to [2], this spinor bundle carries a positive-definite inner product denoted by $\langle\ ,\ \rangle$ and such that

$$\langle \widetilde{\gamma}(X)\varphi, \psi\rangle = -\langle \varphi, \widetilde{\gamma}(X)\psi\rangle \qquad \text{and} \qquad \langle \widetilde{\gamma}(\xi)\varphi, \psi\rangle = \langle \varphi, \widetilde{\gamma}(\xi)\psi\rangle \tag{3}$$

for all $X \in \Gamma(T\Omega)$ and $\varphi, \psi \in \Gamma\big(\mathbb{S}\!\!\!/\Omega\big)$. Moreover, they also satisfy the compatibility relation