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Combinatorics/Game theory

A characterization of b_e-critical trees

Une caractérisation des arbres b_e-critiques

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A B S T R A C T

The b-chromatic number of a graph G is the largest integer k such that G admits a proper coloring with k colors for which each color class contains a vertex that has at least one neighbor in all the other $k-1$ color classes. A graph G is called *b_e-critical* if the contraction of any edge e of G decreases the b-chromatic number of G. The purpose of this paper is the characterization of all *b_e-critical trees*.

R É S U M É

Le nombre b-chromatique d'un graphe G est le plus grand entier k tel que G admette une coloration propre avec k couleurs, pour laquelle toute classe de couleur contient un sommet qui a au moins un voisin dans toutes les autres $k-1$ classes de couleur. Un graphe G est appelé *b_e-critique* si la contraction de toute arête e de G fait diminuer le nombre b-chromatique de G. Le but de cet article est la caractérisation de tous les arbres *b_e-critiques*.

1. Introduction

All graphs in this paper are finite and simple. For the terminology and the notations not defined here we refer to [2]. Let $G = (V(G), E(G))$ be a graph. For a non-empty set $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of G induced by A, and by $G \setminus A$ the subgraph induced by $V(G) \setminus A$. If $A = \{v\}$ we may write $G \setminus v$ instead of $G \setminus \{v\}$. For a vertex v of G, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the degree of v, denoted by $d_G(v)$, is $|N_G(v)|$. By $\Delta(G)$ and $d_G(u, v)$, we denote the *maximum degree* of the graph G and the *distance* between u and v in G, respectively. A *tree* is a connected graph without induced cycle. A *rooted tree* is a tree with a special vertex, called the *root* of the tree. A vertex of degree one is called a *leaf*, and its neighbor is called a *support* vertex. An edge incident with a leaf is called a *pendant edge*.

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A tree T is a *double star* $S_{p,q}$ ($p \geq q \geq 1$) if it contains exactly two vertices x, y (called central vertices) that are not leaves such that $d_T(x) = p + 1$ and $d_T(y) = q + 1$. We let P_n and $K_{1,n-1}$ denote the *path* and *star* on n vertices, respectively.

A *proper coloring* of G is an assignment of colors (represented by natural numbers) to the vertices of G such that any two adjacent vertices have different colors. The minimum number $\chi(G)$ for which there exists a proper coloring (with $\chi(G)$ colors) is called the *chromatic number* of a graph G. A *b-coloring* of a graph by k colors is a proper coloring with the property that each color class contains a vertex that has at least one neighbor in all the other $k-1$ color classes. We call any such vertex a *b-vertex*. The *b-chromatic number* $b(G)$ of a graph G is the largest number k such that G has a b-coloring with k colors. This parameter has been defined by Irving and Manlove [7,10]. It is obvious that $\chi(G) \leq b(G) \leq \Delta(G) + 1$. For arbitrary graphs, the problem of determining $b(G)$ is NP-complete [7,10], even when restricted to bipartite graphs [9]. For the special case of trees, Irving and Manlove [7,10] presented a linear time algorithm. A recent survey on the b-coloring in graphs can be found in [8].

It was observed in [7,10] that if a graph G admits a b-coloring with ℓ colors, G must have at least ℓ vertices with degree at least $\ell - 1$. The *m-degree* of a graph G, denoted $m(G)$, is the largest integer ℓ such that G has ℓ vertices of degree at least $\ell - 1$. Clearly, $m(G) \leq \Delta(G) + 1$. Irving and Manlove [7,10] show that this parameter bounds the b-chromatic number. So, every graph satisfies $b(G) \leq m(G)$. A vertex of G with degree at least $m(G) - 1$ is called a *dense vertex*. A *pivoted tree* is a tree T in which one vertex v of degree less than $m(G) - 1$ is distinguished and called the *pivot*.

Definition 1. [7,10] A tree T is pivoted if T has exactly $m(T)$ dense vertices and T contains a vertex v such that v is not dense and every dense vertex is adjacent either to v or to a neighbor of v of degree $m(T) - 1$.

The following observation is straightforward.

Observation 2. *Every non-dense vertex of a pivoted tree T, except the pivot, may be adjacent to at most one dense vertex of T.*

D.F. Manlove and R.W. Iring [7,10] have proved that, for trees, the b-chromatic number can be computed as follows.

Theorem 3. *[7] If T is a pivoted tree, then $b(T) = m(T) - 1$; else, $b(T) = m(T)$.*

The concept of critical graphs with respect to the b-chromatic number has received more attention in recent years. The graphs for which the b-chromatic number decreases on the deletion of any edge were first studied in [4,6]. Further, a characterization of all such graphs is given in [1]. On the other hand, the authors of [3] characterized the trees whose b-chromatic number decreases when any vertex is removed. The graphs for which the b-chromatic number increases upon the removal of any edge (or vertex) were explored in [5].

In this paper, we study those graphs where the b-chromatic number decreases on the contraction of any edge. Before stating our results, we need some definitions and notation. For a given graph G, the *contraction* of an edge $e = uv$ means removing u and v from the vertex-set $V(G)$ and replacing it by a new vertex z and attaching z to all vertices that are adjacent to u or v in G. We denote by G_e the graph obtained from G by contracting the edge e.

Definition 4. A graph is called b_e-critical if the b-chromatic number decreases upon the contraction of any edge.

More precisely, we say that a graph G is b_e-critical if $b(G_e) < b(G)$ holds for every edge e in G. The aim of the paper is to characterize all b_e-critical trees.

2. Preliminary results

This section presents some results that will be useful in the characterization of b_e-critical trees.

Observation 5. *Let e be an edge of a b_e-critical tree T and let T_e be the tree obtained from T by contracting e. Then,*

(*i*) $m(T_e) \leq m(T)$, *with equality if e is a non-pendant edge such that one of its endpoints is a non-dense vertex.*
(*ii*) *If T_e is not a pivoted tree, then $m(T_e) \leq m(T) - 1$.*

Proof. (*i*) If the first part is not true, then Theorem 3 yields $b(T_e) \geq m(T_e) - 1 \geq m(T) \geq b(T)$, which is a contradiction. The second part follows immediately because contracting such edge does not decrease the m-degree of T.

(*ii*) Using again Theorem 3, we get $m(T_e) = b(T_e) \leq b(T) - 1 = m(T) - 1$. $\square$

For the remainder of this paper, we denote by D and L, respectively, the set of dense vertices and the set of leaves in T. Denote also by D_e and L_e, respectively, the set of dense vertices and the set of leaves in T_e.