C. R. Acad. Sci. Paris, Ser. I 356 (2018) 345–350

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical problems in mechanics

Motion of an incompressible solid with large deformations

Solide incompressible en grande déformation

Elena Bonetti [a,b], Michel Frémond [c]

[a] *Laboratorio Lagrange, Dipartimento di Matematica "Federigo Enriques", Università di Milano, Via Saldini, 50, 20133 Milano, Italy*
[b] *IMATI–CNR, Via Ferrata 1, 27100, Pavia, Italy*
[c] *Laboratorio Lagrange, Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico, 1, 00163 Roma, Italy*

ARTICLE INFO

Article history:
Received 13 October 2017
Accepted 23 January 2018
Available online 14 February 2018

Presented by Philippe G. Ciarlet

ABSTRACT

We study the motion of a visco-elastic solid with large deformations. We prove the existence of a local-in-time motion and of a non-negative pressure, which is a measure reaction to the incompressibility condition.

RÉSUMÉ

On étudie le mouvement d'un solide viscoélastique incompressible en grande déformation. On démontre l'existence d'un mouvement local en temps et d'une pression positive qui est une mesure, réaction à la condition d'incompressibilité.

Version française abrégée

Nous étudions le mouvement d'un solide viscoélastique incompressible en grande déformation. La condition d'incompressibilité (2) est unilatérale, car en traction des vides peuvent apparaître. Cette condition introduit une pression positive (voir (5) et (17)). Par une approximation de Moreau–Yosida de la fonction indicatrice de l'ensemble des matrices d'élongation qui vérifient la condition d'incompressibilité (2), on démontre l'existence d'un mouvement approché. On montre que ce mouvement approché a une limite et que la pression approchée a aussi une limite, qui est une mesure (Theorème 5.1). Cette mesure autorise des collisions, c'est-à-dire des discontinuités de vitesse, lors de la disparition de vides.

1. Introduction

We consider the motion between time 0 and time $\tilde{t} > 0$ of a solid located in a smooth bounded domain $\mathcal{D}_a \subset \mathbb{R}^3$. The position function is $a \in \mathcal{D}_a,\ t \in (0, \tilde{t}) \to \Phi(a, t)$, with $\Phi(a, 0) = a$. For the sake of simplicity, we assume that the solid is in

E-mail addresses: elena.bonetti@unimi.it (E. Bonetti), michel.fremond@lagrange.it (M. Frémond).

https://doi.org/10.1016/j.crma.2018.01.016
1631-073X/

contact on a smooth part Γ_0^a of its boundary with an obstacle schematized by springs applying actions proportional to the gap $(\Phi - a)$ and its gradient. Besides the body force $\overrightarrow{f}$, no other external action is applied.

We denote by $\mathcal{M}$ the space of 3×3 matrices, endowed with the usual scalar product. The subspaces $\mathcal{S} \subset \mathcal{M}$ of the symmetric matrices and $\mathcal{A} \subset \mathcal{M}$ of the antisymmetric matrices are orthogonal. We introduce the set

$$C_\alpha = \left\{ \mathbf{B} \in \mathcal{M} \;\middle|\; \text{tr}\,\mathbf{B} \geq 3\alpha,\ \text{tr}(\text{cof}\,\mathbf{B}) \geq 3\alpha^2,\ \det \mathbf{B} \geq \alpha^3 \right\},\ \ 0 < \alpha < 1, \tag{1}$$

where α is the only physical parameter, the value of which we choose different from 1. We recall that for any position Φ that is kinematically admissible, i.e. differentiable with $\det(\text{grad}\,\Phi) > 0$, there exists a unique symmetric positive definite matrix $\mathbf{W}$, the stretch matrix, and a rotation matrix $\mathbf{R}$ with $\det \mathbf{R} = 1$, such that $\text{grad}\,\Phi = \mathbf{RW}$. With this decomposition, the local impenetrability condition is to require, for the stretch matrix, that $\mathbf{W} \in C_\alpha \cap \mathcal{S}$. Note in particular that the physical constant α quantifies the resistance of the material to crushing.

This model has been introduced in [1], [3], [6], and in [2] (with more analytical details concerning the existence of solutions). We refer the reader to these papers for further details in the derivation of the model and, for some auxiliary results, we will exploit in the sequel. Actually, let us point out that the main novelty of this paper consists in the fact that incompressibility is required as an unilateral internal constraint (see Sec. 2).

2. The incompressibility condition

The usual incompressibility condition is $\det \mathbf{W} = 1$. But let us consider experiments and remark that when tension is applied to a sample, some voids may appear during the evolution, mainly at the microscopic level, with a volume increase at the macroscopic level. Moreover a phase change may occur and eventually makes possible an increase of volume. This behaviour has been described a long time ago by Jean-Jacques Moreau to investigate cavitation in fluid mechanics, [7]. The water is incompressible, but bubbles may appear inside water at the microscopic level when pressure is null: this is the cavitation phenomenon responsible for water hammers. It results that the unilateral condition

$$\det \mathbf{W} \geq 1 \tag{2}$$

is possible. On the contrary, for an incompressible material, it is impossible to have interpenetration at the microscopic level. It results $\det \mathbf{W} < 1$ is impossible. Note that the word incompressible refers to the impossibility to modify the volume by compression. We are motivated to think that condition (2) is the condition that accounts for the actual mechanical behaviour. The set

$$K = \{ \mathbf{B} \in \mathcal{M} \mid \mathbf{B} \in \mathcal{S} \cap C_0,\ \det \mathbf{B} \geq 1 \} \tag{3}$$

is convex (C_0 is set C_α with $\alpha = 0$, $\mathcal{S} \cap C_0$ is the set of the semi-definite matrices). We denote by I_K the indicator function of set K in $\mathcal{M}$. Set K accounts for the two internal constraints: symmetry of stretch matrix and incompressibility.

3. The constitutive laws

We derive the constitutive laws from volume free energy $\Psi(\mathbf{W}, \text{grad}\Delta\Phi, \|\text{grad}\mathbf{R}\|^2)$, surface free energy $\Psi_\Gamma(\Phi - a, \text{grad}\Phi - \mathbf{I})$ and volume pseudo-potential of dissipation $D(\dot{\mathbf{W}}, \text{grad}\,\Omega)$ with $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ and

$$\Psi(\mathbf{W}, \text{grad}\ \Delta\Phi, \|\text{grad}\,\mathbf{R}\|^2) = \frac{1}{2}\|\mathbf{W} - \mathbf{I}\|^2 + \frac{1}{2}\|\text{grad}\ \Delta\Phi\|^2 + \hat{\Psi}(\mathbf{W}) + I_K(\mathbf{W}) + \frac{1}{4}\|\text{grad}\,\mathbf{R}\|^2,$$

$$\Psi_\Gamma(\Phi - a, \text{grad}\ \Phi - \mathbf{I}) = \frac{1}{2}\int_{\Gamma_0^a} (\Phi - a)^2 \text{d}\Gamma + \frac{1}{2}\int_{\Gamma_0^a} (\text{grad}\ \Phi - \mathbf{I})^2 \text{d}\Gamma,$$

and

$$D(\dot{\mathbf{W}}, \text{grad}\,\Omega) = \frac{1}{2}\|\dot{\mathbf{W}}\|^2 + \frac{1}{4}\|\text{grad}\,\Omega\|^2,$$

where $\mathbf{W}$ is a matrix of $\mathcal{M}$, and $\|\mathbf{W}\|^2 = \mathbf{W} : \mathbf{W}$, $\|\text{grad}\ \Delta\Phi\|^2 = \Phi_{i,\alpha\beta\beta}\Phi_{i,\alpha\delta\delta}$. The function $I_K(\mathbf{W})$ is the indicator function of convex set K that insures the symmetry of matrix $\mathbf{W}$ and incompressibility.

The free energy accounts for the impenetrability condition. In particular, this constraint is related to the presence of function $\hat{\Psi}(\mathbf{W})$, which is a smooth approximation from the interior of the indicator function of the set C_α in $\mathcal{M}$ (see [1], [2], [3] and [6]).

The incompressibility constitutive law is

$$\mathbf{S}_{\text{reac}} + \mathbf{A}_{\text{reac}} \in \partial I_K(\mathbf{W}),$$