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Number theory

On the denominators of harmonic numbers ☆

Sur les dénominateurs des nombres harmoniques

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ABSTRACT

Let H_n be the n-th harmonic number and let v_n be its denominator. It is well known that v_n is even for every integer $n \geq 2$. In this paper, we study the properties of v_n. One of our results is: the set of positive integers n such that v_n is divisible by the least common multiple of $1, 2, \cdots, \lfloor n^{1/4} \rfloor$ has density one. In particular, for any positive integer m, the set of positive integers n such that v_n is divisible by m has density one.

RÉSUMÉ

Soit H_n le n-ième nombre harmonique et notons v_n son dénominateur. Il est bien connu que v_n est pair pour tout entier $n \geq 2$. Dans ce texte, nous étudions les propriétés de v_n. Un de nos résultats montre que l'ensemble des entiers positifs n tels que v_n soit divisible par le plus petit commun multiple de $1, 2, \ldots, [n^{1/4}]$ est de densité 1. En particulier, pour tout entier positif m, l'ensemble des entiers positifs n tels que v_n soit divisible par m est de densité 1.

1. Introduction

For any positive integer n, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{u_n}{v_n}, \quad (u_n, v_n) = 1, \ v_n > 0.$$

The number H_n is called *the n-th harmonic number*. In 1991, Eswarathasan and Levine [2] introduced I_p and J_p. For any prime number p, let J_p be the set of positive integers n such that $p \mid u_n$ and let I_p be the set of positive integers n such that $p \nmid v_n$. Here I_p and J_p are slightly different from those in [2]. In [2], Eswarathasan and Levine considered $0 \in I_p$ and $0 \in J_p$. It is clear that $J_p \subseteq I_p$.

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In 1991, Eswarathasan and Levine [2] conjectured that J_p is finite for any prime number p. In 1994, Boyd [1] confirmed that J_p is finite for $p \le 547$, except $83, 127, 397$. For any set S of positive integers, let $S(x) = |S \cap [1, x]|$. In 2016, Sanna [3] proved that

$$J_p(x) \le 129\, p^{\frac{2}{3}}\, x^{0.765}.$$

Recently, Wu and Chen [5] proved that

$$J_p(x) \le 3\, x^{\frac{2}{3} + \frac{1}{25 \log p}}. \tag{1.1}$$

For v_n, Shiu [4] proved that, for any primes $2 < p_1 < p_2 < \cdots < p_k$, there exists n such that the least common multiple of $1, 2, \cdots, n$ is divisible by $p_1 \cdots p_k v_n$.

For any positive integer m, let I_m be the set of positive integers n such that $m \nmid v_n$. In this paper, the following results are proved.

Theorem 1.1. *The set of positive integers n such that v_n is divisible by the least common multiple of $1, 2, \cdots, \lfloor n^{1/4} \rfloor$ has density one.*

Theorem 1.2. *For any positive integer m and any positive real number x, we have*

$$I_m(x) \le 4\; m^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q_m}},$$

where q_m is the least prime factor of m.

From Theorem 1.1 or Theorem 1.2, we immediately have the following corollary.

Corollary 1.3. *For any positive integer m, the set of positive integers n such that $m \mid v_n$ has density one.*

2. Proofs

We always use p to denote a prime. Firstly, we give the following two lemmas.

Lemma 2.1. *For any prime p and any positive integer k, we have*

$$I_{p^k} = \{p^k n_1 + r : n_1 \in J_p \cup \{0\},\ 0 \le r \le p^k - 1\} \setminus \{0\}.$$

Proof. For any integer a, let $\nu_p(a)$ be the p-adic valuation of a. For any rational number $\alpha = \frac{a}{b}$, let $\nu_p(\alpha) = \nu_p(a) - \nu_p(b)$. It is clear that $n \in I_{p^k}$ if and only if $\nu_p(H_n) > -k$.

If $n < p^k$, then $\nu_p(H_n) \ge -\nu_p([1, 2, \cdots, n]) > -k$. So $n \in I_{p^k}$. In the following, we assume that $n \ge p^k$. Let

$$n = p^k n_1 + r, \quad 0 \le r \le p^k - 1,\ n_1, r \in \mathbb{Z}.$$

Then $n_1 \ge 1$. Write

$$H_n = \sum_{m=1, p^k \nmid m}^{n} \frac{1}{m} + \frac{1}{p^k} H_{n_1} = \frac{b}{p^{k-1} a} + \frac{u_{n_1}}{p^k v_{n_1}} = \frac{p b v_{n_1} + a u_{n_1}}{p^k a v_{n_1}}, \tag{2.1}$$

where $p \nmid a$ and $(u_{n_1}, v_{n_1}) = 1$.

If $n_1 \in J_p$, then $p \mid u_{n_1}$ and $p \nmid v_{n_1}$. Thus $p \mid a u_{n_1} + p b v_{n_1}$ and $\nu_p(p^k a v_{n_1}) = k$. By (2.1), $\nu_p(H_n) > -k$. So $n \in I_{p^k}$.

If $n_1 \notin J_p$, then $p \nmid u_{n_1}$. Thus $p \nmid a u_{n_1} + p b v_{n_1}$. It follows from (2.1) that $\nu_p(H_n) \le -k$. So $n \notin I_{p^k}$.

Now we have proved that $n \in I_{p^k}$ if and only if $n_1 \in J_p \cup \{0\}$.

This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *For any prime power p^k and any positive number x, we have*

$$I_{p^k}(x) \le 4 (p^k)^{\frac{1}{3} - \frac{1}{25 \log p}} x^{\frac{2}{3} + \frac{1}{25 \log p}}.$$

Proof. If $x \le p^k$, then

$$I_{p^k}(x) \le x < 4 x^{\frac{1}{3} - \frac{1}{25 \log p}} x^{\frac{2}{3} + \frac{1}{25 \log p}} \le 4 (p^k)^{\frac{1}{3} - \frac{1}{25 \log p}} x^{\frac{2}{3} + \frac{1}{25 \log p}}.$$