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Homological algebra/Differential geometry

Formality theorem for differential graded manifolds ☆

Théorème de formalité pour les variétés différentielles graduées

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A R T I C L E I N F O

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A B S T R A C T

We establish a formality theorem for smooth dg manifolds. More precisely, we prove that, for any finite-dimensional dg manifold $(\mathcal{M}, Q)$, there exists an L_∞ quasi-isomorphism of dglas from $\left({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [Q, -], [-, -]\right)$ to $\left({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [\![m + Q, -]\!], [\![-, -]\!]\right)$ whose first Taylor coefficient (1) is equal to the composition $\mathrm{hkr} \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} : {}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}) \to {}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ of the action of $(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} \left(\Omega^k(\mathcal{M})\right)^k$ on ${}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ (by contraction) with the Hochschild–Kostant–Rosenberg map and (2) preserves the associative algebra structures on the level of cohomology. As an application, we prove the Kontsevich–Shoikhet conjecture: a Kontsevich–Duflo-type theorem holds for all finite-dimensional smooth dg manifolds.

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R É S U M É

Nous prouvons un théorème de formalité pour les variétés lisses différentielles graduées. Plus précisément, nous prouvons qu'il existe, pour toute variété différentielle graduée $(\mathcal{M}, Q)$, un quasi-isomorphisme L_∞ de l'algèbre de Lie différentielle graduée $\left({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [Q, -], [-, -]\right)$ dans l'algèbre de Lie différentielle graduée $\left({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [\![m + Q, -]\!], [\![-, -]\!]\right)$, dont le premier coefficient de Taylor (1) est égal à la composée $\mathrm{hkr} \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} : {}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}) \to {}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ de l'action (par contraction) de $(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} \left(\Omega^k(\mathcal{M})\right)^k$ sur ${}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ avec l'application de Hochschild–Kostant–Rosenberg et (2) respecte les structures d'algèbres associatives en cohomologie. Comme application, nous prouvons la conjecture de Kontsevich–Shoikhet : il existe un théorème de type Kontsevich–Duflo valable pour toute variété différentielle graduée de dimension finie.

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1. Introduction

In 1997, Kontsevich revolutionized the field of deformation quantization with his formality theorem: there exists an L_∞ quasi-isomorphism from the dgla $T_{\mathrm{poly}}(M)$ of polyvector fields on a smooth manifold M to the dgla $D_{\mathrm{poly}}(M)$ of polydifferential operators on M whose first "Taylor coefficient" is the classical Hochschild–Kostant–Rosenberg map [19]. Kontsevich's formality theorem completely settled a long-standing problem [3] regarding the existence and classification of deformation quantizations for all smooth Poisson manifolds. An alternative approach to the formality theorem was developed by Tamarkin using operads [31].

Beyond deformation quantization, Kontsevich's formality construction found other important applications in several different areas of mathematics. One of them is the extension of the classical Duflo theorem. Given a finite-dimensional Lie algebra $\mathfrak{g}$, the Poincaré–Birkhoff–Witt (PBW) map is the isomorphism of $\mathfrak{g}$-modules $\mathrm{pbw}: S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})$ defined by the symmetrization map $X_1 \odot \cdots \odot X_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n} X_{\sigma(1)}\cdots X_{\sigma(n)}$. It induces an isomorphism $\mathrm{pbw}: S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ between subspaces of $\mathfrak{g}$-invariants. This isomorphism fails to intertwine the obvious multiplications on $S(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$. Nevertheless, it can be modified so as to become an isomorphism of associative algebras. The Duflo element $J \in \widehat{S}(\mathfrak{g}^\vee)$ is the formal power series on $\mathfrak{g}$ defined by $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$, for all $x \in \mathfrak{g}$. Considered as a formal linear differential operator on $\mathfrak{g}^\vee$ with constant coefficients, the square root of the Duflo element defines a transformation $J^{\frac{1}{2}} : S(\mathfrak{g}) \to S(\mathfrak{g})$. A remarkable theorem due to Duflo [14] asserts that the composition $\mathrm{pbw}\circ J^{\frac{1}{2}} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. Duflo's original proof is based on deep and sophisticated techniques of representation theory including Kirillov's orbit method. As an application of his formality construction, Kontsevich proposed a new proof of Duflo's theorem by means of the associative algebra structure carried by the tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich's approach [19] led to an extension of Duflo's theorem: for every finite dimensional Lie algebra $\mathfrak{g}$, the map $\mathrm{pbw}\circ J^{\frac{1}{2}} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ is an isomorphism of graded associative algebras. The classical Duflo theorem is simply the isomorphism between the cohomology groups of degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [29] (see also [22,23]). Furthermore, Kontsevich discovered a similar phenomenon in complex geometry [19]. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of a complex manifold X are defined as the groups $\mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Gerstenhaber–Shack [18] derived an isomorphism of cohomology groups $\mathrm{hkr}: H^\bullet(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$ from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications on the two cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [19] obtained the following theorem: the composition $\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$, where Td_X denotes the Todd class of the complex manifold X, is an isomorphism of associative algebras. The multiplications on $H^\bullet(X, \Lambda^\bullet T_X)$ and $HH^\bullet(X)$ are respectively the wedge product and the Yoneda product. Calaque–Van den Bergh [6] wrote a detailed proof of Kontsevich's theorem and showed additionally that the map $\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}}$ actually respects the Gerstenhaber algebra structures carried by the two cohomologies. A related result was also proved by Dolgushev–Tamarkin–Tsygan [12,13].

Hence Kontsevich's formality construction revealed a hidden connection between complex geometry and Lie theory. Kontsevich's discovery of this mysterious and surprising similarity between the Todd class of a complex manifold and the Duflo element of a Lie algebra — two seemingly unrelated objects — was responsible for many subsequent exciting developments. Naturally, one would wonder whether a general framework encompassing both Lie algebras and complex manifolds as special cases could be developed in which a Kontsevich–Duflo-type theorem would hold. This is indeed the main goal of this Note. We claim that *differential graded (dg) manifolds* provide the appropriate framework.

By a dg manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field Q of degree $+1$ satisfying $[Q,Q]=0$. Dg manifolds arise naturally in many situations in geometry, Lie theory, and mathematical physics. Standard examples of dg manifolds are: (1) *Lie algebras* — Given a finite-dimensional Lie algebra $\mathfrak{g}$, we write $\mathfrak{g}[1]$ to denote the dg manifold having $C^\infty(\mathfrak{g}[1]) = \wedge^\bullet \mathfrak{g}^\vee$ as its algebra of functions and the Chevalley–Eilenberg differential $Q = d_{\mathrm{CE}}$ as its homological vector field. This construction admits an up-to-homotopy version: given a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ of finite type (i.e. each $\mathfrak{g}_i$ is a finite-dimensional vector space), $\mathfrak{g}[1]$ is a dg-manifold if and only if $\mathfrak{g}$ is a curved L_∞ algebra. (2) *Complex manifolds* — Given a complex manifold X, we write $T^{0,1}_X[1]$ to denote the dg manifold having $C^\infty(T^{0,1}_X[1]) \cong \Omega^{0,\bullet}(X)$ as its algebra of functions and the Dolbeault operator $Q = \bar\partial$ as its homological vector field. (3) *Derived intersections* — Given a smooth section s of a smooth vector bundle $E \to M$, we write $E[-1]$ to denote the dg-manifold having $C^\infty(E[-1]) = \Gamma(\wedge^{-\bullet}(E^\vee))$ as its algebra of functions and the contraction operator i_s as its homological vector field.

In 1998, Shoikhet [30] proposed a conjecture, known as *Kontsevich–Shoikhet conjecture*, stating that a Kontsevich–Duflo-type formula holds for all finite-dimensional smooth dg manifolds. In this Note, we prove a formality theorem for smooth dg manifolds (Theorem 4.2) and, as an immediate consequence, we confirm the Kontsevich–Shoikhet conjecture (Theorem 4.3). Applying Theorem 4.3 to the dg manifold examples of type (1) and (2) mentioned earlier, we recover the Kontsevich–Duflo theorem for Lie algebras and Kontsevich's theorem for complex manifolds, respectively. Thus we fulfill our stated

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