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Remarks on the Monge–Kantorovich problem in the discrete setting

Remarques sur le problème de Monge–Kantorovich dans le cas discret

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A R T I C L E I N F O

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A B S T R A C T

In Optimal Transport theory, three quantities play a central role: the minimal cost of transport, originally introduced by Monge, its relaxed version introduced by Kantorovich, and a dual formulation also due to Kantorovich. The goal of this Note is to publicize a very elementary, self-contained argument extracted from [9], which shows that all three quantities coincide in the discrete case.

R É S U M É

En théorie du transport optimal, trois quantités jouent un rôle central : le coût minimal de transport, introduit par Monge, sa version relaxée, introduite par Kantorovich, et la formulation duale, due aussi à Kantorovich. L'objet de cette note est de mettre en avant une démonstration totalement élémentaire, extraite de [9], du fait que ces trois quantités coïncident dans le cas discret ; cette preuve ne requiert aucune connaissance préalable.

1. Introduction

Consider two sets X, Y consisting of m points (P_i) and (N_i), $1 \le i \le m$, i.e.

$$X = \{P_i, P_2, \dots, P_m\} \text{ and } Y = \{N_1, N_2, \dots, N_m\}.$$

Let $c : X \times Y \to \mathbb{R}$ be any function (c stands for "cost"). We introduce three quantities. The first one denoted M (for Monge) is defined by

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$$M := \min_{\sigma \in \mathcal{S}_m} \sum_{i=1}^{m} c(P_i, N_{\sigma(i)}), \tag{1}$$

where the minimum is taken over the set $\mathcal{S}_m$ of all permutations of the integers $\{1, 2, \dots, m\}$. The second one, denoted K (for Kantorovich), is defined by

$$K := \min_A \left\{ \sum_{i,j=1}^{m} a_{ij} c(P_i, N_j);\ A = (a_{ij}) \text{ is doubly stochastic} \right\}. \tag{2}$$

Recall that a matrix $A = (a_{ij})$ is doubly stochastic if

$$a_{ij} \geq 0\, \forall i, j, \ \sum_{i=1}^{m} a_{ij} = 1\ \forall j, \ \text{ and } \sum_{j=1}^{m} a_{ij} = 1\ \forall i. \tag{3}$$

Finally define D (for duality) by

$$D := \sup_{\substack{\psi: Y \to \mathbb{R} \\ \varphi: X \to \mathbb{R}}} \left\{ \sum_{i=1}^{m} (\varphi(P_i) - \psi(N_i));\ \varphi(x) - \psi(y) \leq c(x, y),\ \forall x \in X, \forall y \in Y \right\}. \tag{4}$$

Theorem 1.1. *We have*

$$M = K = D. \tag{5}$$

Moreover the "sup" in (4) is achieved.

Equality $K = D$ in Theorem 1.1 is at the heart of Kantorovich's pioneering discovery concerning the Monge problem (see [19] and [20]). Equality $M = K$ makes totally transparent the connection between Kantorovich's formulation and Monge's original goal (see item (3) in Section 4 below). The purpose of this note is to advertise the MK (= Monge–Kantorovich) theory in its most elementary (but in itself striking and useful!) setting, as it appears, e.g., in Brezis–Coron–Lieb [11] (see Section 3 and item (1) in Section 4 below). This "primitive" case illuminates the foundations of the MK saga which has "exploded" in recent years; see, e.g., the remarkable works of [2], [3], [7], [15], [16], [17], [23], [24], [29], [34], [35], etc. I reproduce in Section 2 an elementary self-contained proof of Theorem 1.1 (accessible to first-year students), extracted from a presentation of [11] that I gave in 1985 (see [9]).

2. Proof of Theorem 1.1

Choosing for A in (2) a permutation matrix yields

$$K \leq M. \tag{6}$$

On the other hand, assume that φ and ψ are as in (4). Let $A = (a_{ij})$ be a doubly stochastic matrix. Multiplying the inequalities $\varphi(P_i) - \psi(N_j) \leq c(P_i, N_j)$ by a_{ij} and summing over i, j yields

$$\sum_{i=1}^{m} (\varphi(P_i) - \psi(N_i)) \leq \sum_{i,j=1}^{m} a_{ij} c(P_i, N_j). \tag{7}$$

Minimizing over A and maximizing over φ, ψ gives

$$D \leq K. \tag{8}$$

In view of (6) and (8), it suffices to establish that

$$M \leq D. \tag{9}$$

Proof of (9). Without loss of generality we may relabel the points (N_j) so that

$$M = \sum_{i=1}^{m} c(P_i, N_i) \leq \sum_{j=1}^{m} c(P_j, N_{\sigma(j)}) \quad \forall \sigma \in \mathcal{S}_m. \tag{10}$$

By (4) it remains to show that there exist functions $\varphi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ such that

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