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Geometry

Compact simple Lie groups admitting left-invariant Einstein metrics that are not geodesic orbit ☆

Groupes de Lie simples compacts admettant des métriques d'Einstein invariantes à gauche, dont une géodésique n'est pas une orbite

Huibin Chen, Zhiqi Chen, Shaoqiang Deng

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China

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A B S T R A C T

In this article, we prove that the compact simple Lie groups $\mathrm{SU}(n)$ for $n \geq 6$, $\mathrm{SO}(n)$ for $n \geq 7$, $\mathrm{Sp}(n)$ for $n \geq 3$, $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, and F_4 admit left-invariant Einstein metrics that are not geodesic orbit. This gives a positive answer to an open problem recently posed by Nikonorov.

R É S U M É

Dans cet article, nous démontrons que les groupes simples compacts $SU(n)$ pour $n \geq 6$, $SO(n)$ pour $n \geq 7$, $Sp(n)$ pour $n \geq 3$, E_6, E_7, E_8 et F_4 admettent des métriques d'Einstein invariantes à gauche, dont une géodésique maximale n'est pas une orbite d'un sous-groupe à un paramètre du groupe des isométries complet. Ceci fournit une réponse positive à un problème récemment posé par Nikonorov.

1. Introduction

The purpose of this short note is to give a positive answer to an open problem recently posed by Nikonorov. In his paper [10], Nikonorov proved that there exists a left-invariant Einstein metric on the compact simple Lie group G_2 that is not a geodesic orbit metric. This metric is the first non-naturally reductive left-invariant Einstein metric on G_2 discovered by I. Chrysikos and Y. Sakane in [6]. Recall that a Riemannian metric on a connected manifold M is said to be a geodesic orbit metric if any maximal geodesic of the metric is the orbit of a one-parameter subgroup of the full group of isometries (in this case, the Riemannian manifold is called a geodesic orbit space). It is well known that any naturally reductive metric must

☆ Supported by NSFC (No. 11671212, 51535008) of China.
E-mail addresses: chenhuibin@mail.nankai.edu.cn (H. Chen), chenzhiqi@nankai.edu.cn (Z. Chen), dengsq@nankai.edu.cn (S. Deng).

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be geodesic orbit, but the converse is not true. Recently, many interesting results have been established on non-naturally reductive homogeneous Einstein metrics. It is therefore a natural problem to study homogeneous Einstein metrics that are not geodesic orbit.

The following problem is posed in [10].

Problem 1.1. Is there any other compact simple Lie group admitting a left-invariant Einstein metric that is not geodesic orbit?

The main result of this short note is the following.

Theorem 1.2. *The compact simple Lie groups* $\mathrm{SU}(n)$ *for* $n \geq 6$, $\mathrm{SO}(n)$ *for* $n \geq 7$, $\mathrm{Sp}(n)$ *for* $n \geq 3$, $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ *and* F_4 *admit left-invariant Einstein metrics that are not geodesic orbit.*

2. Some known results and generalized Wallach spaces

In this section, we will recall a sufficient and necessary condition in [10] for a left-invariant metric on a compact Lie group to be a geodesic orbit metric. We will also give some results on generalized Wallach spaces. We first recall a result of [7] on the characterization of geodesic orbit metrics.

Lemma 2.1 *([7]). Let M be a homogeneous Riemannian manifold and G the identity component of the full group of isometries. Write $M = G/H$, where H is the isotropic subgroup of G at $x \in M$, and suppose the Lie algebra of G has a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}\, H$, and $\mathfrak{m}$ is the orthogonal complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an $\mathrm{Ad}H$-invariant inner product on $\mathfrak{g}$. Then M is a geodesic orbit space if and only if, for any $X \in \mathfrak{m}$, there exits $Z \in \mathfrak{h}$ such that $([X+Z, Y]_{\mathfrak{m}}, X) = 0$ for all $Y \in \mathfrak{m}$.

In [10], the author obtained a sufficient and necessary condition for a left-invariant Riemannian metric on a compact Lie group to be a geodesic orbit metric.

Theorem 2.2 *([10]). A simple compact Lie group G with a left-invariant Riemannian metric ρ is a geodesic orbit space if and only if there is a closed connected subgroup K of G such that for any $X \in \mathfrak{g}$ there exists $W \in \mathfrak{k}$ such that for any $Y \in \mathfrak{g}$ the equality $([X+W, Y], X) = 0$ holds or, equivalently, $[A(X), X+W] = 0$, where $A : \mathfrak{g} \to \mathfrak{g}$ is a metric endomorphism and $\mathfrak{g}, \mathfrak{k}$ are the Lie algebras of Lie groups G, K, respectively.*

We now recall the definition of generalized Wallach spaces. Let G/K be a reductive homogeneous space, where G is a semi-simple compact connected Lie group, K is a connected closed subgroup of G, and $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras, respectively. If $\mathfrak{m}$, the tangent space of G/K at $o = \pi(e)$, can be decomposed into three $\mathrm{ad}(\mathfrak{k})$-invariant irreducible summands pairwise orthogonal with respect to B as:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

such that $[\mathfrak{m}_i, \mathfrak{m}_i] \in \mathfrak{k}$ for $i \in \{1,2,3\}$ and $[\mathfrak{m}_i, \mathfrak{m}_j] \in \mathfrak{m}_k$ for $\{i,j,k\} = \{1,2,3\}$, then G/K is called a generalized Wallach space.

In [5] and [10], the authors gave a complete classification of generalized Wallach spaces with G simple. Based on this result, the authors in [3] obtained some Einstein metrics arising from generalized Wallach spaces. We now recall some results in [3].

Let $\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{m}_{p+1} \oplus \mathfrak{m}_{p+2} \oplus \mathfrak{m}_{p+3} = (\mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p) \oplus (\mathfrak{k}_{p+1} \oplus \mathfrak{k}_{p+2} \oplus \mathfrak{k}_{p+3})$. We assume that $\dim_{\mathbb{R}} \mathfrak{k}_0 \leq 1$ and the ideals $\mathfrak{k}_i$ are mutually non-isomorphic for $i = 1, \cdots, p$. We consider the following inner product on $\mathfrak{g}$:

$$\langle\ ,\ \rangle = u_0 B(\ ,\ )|_{\mathfrak{k}_0} + \cdots + u_p B(\ ,\ )|_{\mathfrak{k}_p} + u_{p+1} B(\ ,\ )|_{\mathfrak{k}_{p+1}} + u_{p+2} B(\ ,\ )|_{\mathfrak{k}_{p+2}} + u_{p+3} B(\ ,\ )|_{\mathfrak{k}_{p+3}}, \tag{2.1}$$

where $B(\ ,\ )$ is the negative Killing form of $\mathfrak{g}$ and $u_i \in \mathbb{R}^+$ for all $0 \neq i \neq p+3$.

Denote $d_i = \dim_{\mathbb{R}} \mathfrak{k}_i$ and let $\{e^i_\alpha\}_{\alpha=1}^{d_i}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$, in the sense that $e^i_\alpha \in \mathfrak{k}_i$ and α is the number of basis in $\mathfrak{k}_i$. Let $A^\gamma_{\alpha,\beta} = B([e^i_\alpha, e^j_\beta], e^k_\gamma)$, equivalently, $A^\gamma_{\alpha,\beta}$ are determined uniquely by the identity $[e^i_\alpha, e^j_\beta] = \sum_\gamma A^\gamma_{\alpha,\beta} e^k_\gamma$. Set

$$(ijk) := \left[\begin{matrix} i \\ j\,k \end{matrix} \right] = \sum (A^\gamma_{\alpha,\beta})^2,$$