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Bounded gaps between primes and the length spectra of arithmetic hyperbolic 3-orbifolds

Petits écarts entre idéaux premiers et spectres de longueurs de 3-variétés hyperboliques arithmétiques

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A R T I C L E I N F O

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A B S T R A C T

In 1992, Reid asked whether hyperbolic 3-manifolds with the same geodesic length spectra are necessarily commensurable. While this is known to be true for arithmetic hyperbolic 3-manifolds, the non-arithmetic case is still open. Building towards a negative answer to this question, Futer and Millichap recently constructed infinitely many pairs of non-commensurable, non-arithmetic hyperbolic 3-manifolds which have the same volume and whose length spectra begin with the same first m geodesic lengths. In the present paper, we show that this phenomenon is surprisingly common in the arithmetic setting. In particular, given any arithmetic hyperbolic 3-orbifold derived from a quaternion algebra, any finite subset S of its geodesic length spectrum, and any $k \geq 2$, we produce infinitely many k-tuples of arithmetic hyperbolic 3-orbifolds which are pairwise non-commensurable, have geodesic length spectra containing S, and have volumes lying in an interval of (universally) bounded length. The main technical ingredient in our proof is a bounded gaps result for prime ideals in number fields lying in Chebotarev sets which extends recent work of Thorner.

R É S U M É

En 1992, Reid a demandé si deux 3-variétés hyperboliques partageant le même spectre de longueurs géodésiques sont nécessairement commensurables. Ceci s'avère être vrai quand les variétés sont arithmétiques, mais la question reste ouverte dans le cas non arithmétique. Comme premier pas vers une réponse négative à cette question, Futer et Millichap ont récemment construit un nombre infini de paires de 3-variétés hyperboliques non arithmétiques et non commensurables ayant le même volume et dont les spectres de longueurs commencent avec les mêmes m longueurs géodésiques. Dans le présent article, nous démontrons que ce phénomène est étonnamment commun dans le contexte arithmétique. En particulier, étant donné une 3-variété hyperbolique arithmétique dérivée d'une algèbre de quaternions, un sous-ensemble fini S de son spectre de longueurs

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géodésiques et un entier $k \geq 2$, nous construisons un nombre infini de k-tuples de 3-variétés hyperboliques arithmétiques qui sont non commensurables deux à deux, ont un spectre de longueurs géodésiques contenant S et dont le volume appartient à un intervalle de longueur bornée (cette borne est, en outre, universelle pour chaque entier k). Notre preuve s'appuie sur un résultat sur les petits écarts entre idéaux premiers d'un corps de nombres appartenant à un ensemble de Chebotarev ; ce résultat généralise un article récent de Thorner.

1. Introduction

Given a closed, negatively curved Riemannian manifold M with fundamental group $\pi_1(M)$, each $\pi_1(M)$-conjugacy class $[\gamma]$ has a unique geodesic representative. The multi-set of lengths of these closed geodesics is called the **geodesic length spectrum** and is denoted by $\mathscr{L}(M)$. The extent to which $\mathscr{L}(M)$ determines M is a basic problem in geometry and is the main topic of the present paper. Specifically, our interest lies with the following question, which was posed and studied by Reid [13,14]:

Question 1. *If M_1, M_2 are complete, orientable, finite volume hyperbolic n-manifolds and $\mathscr{L}(M_1) = \mathscr{L}(M_2)$, then are M_1, M_2 commensurable?*

The motivation for this question is two-fold. First, Reid [13] gave an affirmative answer to Question 1 when $n = 2$ and M_1 is arithmetic. In particular, if M_1 is arithmetic and $\mathscr{L}(M_1) = \mathscr{L}(M_2)$, then M_1, M_2 are commensurable and hence M_2 is also arithmetic as arithmeticity is a commensurability invariant. Second, the two most common constructions of Riemannian manifolds with the same geodesic length spectra (Sunada [15], Vignéras [17]) both produce manifolds that are commensurable. Question 1 has been extensively studied in the arithmetic setting (i.e., when M_1 is arithmetic). When $n = 3$, Chinburg–Hamilton–Long–Reid [3] gave an affirmative answer. Prasad–Rapinchuk [12] later showed that the geodesic length spectrum of an arithmetic hyperbolic n-manifold determines the manifold up to commensurability when $n \not\equiv 1 \pmod 4$ and $n \neq 7$. Most recently, Garibaldi [5] has confirmed the question in dimension $n = 7$.

In the non-arithmetic setting (i.e., when neither M_1 nor M_2 is arithmetic), the relationship between the geodesic length spectrum and commensurability class of the manifold is rather mysterious. To our knowledge, the only explicit work in this area is Millichap [11] and Futer–Millichap [4]. In [4], which extends work from [11], Futer and Millichap produce, for every $m \geq 1$, infinitely many pairs of non-commensurable hyperbolic 3-manifolds which have the same volume and the same m shortest geodesic lengths. Additionally, they give an upper bound on the volume of their manifolds as a function of m. In this paper we also consider hyperbolic 3-manifolds and orbifolds. Note that in this context we consider the complex length spectrum, which encodes both the real length of a closed geodesic as well as the holonomy angle incurred in traveling once around the geodesic. Inspired by [4], in this paper we consider the following question.

Question 2. *Let M be an arithmetic hyperbolic 3-orbifold and S be a finite subset of the complex length spectrum $\mathscr{L}(M)$ of M. What can one say about the set of hyperbolic 3-orbifolds N which are not commensurable with M and for which $\mathscr{L}(N)$ contains S?*

This question was previously studied by the authors in [8]. Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 3-orbifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S. In [8], it was shown that, if $\pi(V, S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V, S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V. This shows that not only is it quite common for an arithmetic hyperbolic 3-orbifold to share large portions of its geodesic length spectrum with other (non-commensurable) arithmetic hyperbolic 3-orbifolds, but that the cardinality of sets of commensurability classes of such orbifolds grows relatively fast.

A few remarks about the hypothesis that $\pi(V, S) \to \infty$ as $V \to \infty$ are in order. In [8] a number field K (containing a unique complex place) and collection of quadratic field extensions $L_1, \ldots, L_r$ of K were associated with S. Theorem 4.10 of [8] shows that a necessary and sufficient condition for $\pi(V, S) \to \infty$ as $V \to \infty$ is that there exist infinitely many quaternion algebras over K which are ramified at all real places of K and which admit embeddings of all of the extensions L_i/K. The Albert–Brauer–Hasse–Noether theorem, which characterizes when a quaternion algebra over a number field admits an embedding of a quadratic extension, therefore implies that it is quite common for $\pi(V, S) \to \infty$ as $V \to \infty$. It is, however, possible for $\pi(V, S)$ to be non-zero yet eventually constant. In light of the comments above, this amounts to constructing