C. R. Acad. Sci. Paris, Ser. I 355 (2017) 1161–1167

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Numerical analysis

An LP empirical quadrature procedure for parametrized functions

Une procédure de quadrature empirique par programmation linéaire pour les fonctions à paramètres

Anthony T. Patera [a], Masayuki Yano [b]

[a] *Room 3-266, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA*
[b] *University of Toronto, 4925 Duffein Street, Toronto, ON, M3H 5T6, Canada*

ARTICLE INFO

Article history:
Received 27 August 2017
Accepted after revision 30 October 2017
Available online 10 November 2017

Presented by Olivier Pironneau

ABSTRACT

We extend the linear program empirical quadrature procedure proposed in [9] and subsequently [3] to the case in which the functions to be integrated are associated with a parametric manifold. We pose a discretized linear semi-infinite program: we minimize as objective the sum of the (positive) quadrature weights, an ℓ_1 norm that yields sparse solutions and furthermore ensures stability; we require as inequality constraints that the integrals of J functions sampled from the parametric manifold are evaluated to accuracy $\bar{\delta}$. We provide an *a priori* error estimate and numerical results that demonstrate that under suitable regularity conditions, the integral of any function from the parametric manifold is evaluated by the empirical quadrature rule to accuracy $\bar{\delta}$ as $J \to \infty$. We present two numerical examples: an inverse Laplace transform; reduced-basis treatment of a nonlinear partial differential equation.

RÉSUMÉ

Nous étendons la procédure de quadrature empirique par programmation linéaire proposée dans [9] et par la suite dans [3] au cas où les fonctions à intégrer sont associées à une variété paramétrique. Nous posons un problème de programmation linéaire discret et semi-infini : nous minimisons la fonction objectif, qui est la somme des poids (positifs) de quadrature, qui constitue une norme ℓ_1 menant à des solutions parcimonieuses et assurant la stabilité, les contraintes d'inégalité requises étant que les intégrales de J fonctions échantillonnées à partir de la variété soient évaluées à une précision $\bar{\delta}$. Nous fournissons un estimateur d'erreur a priori et des résultats numériques qui démontrent que, sous certaines conditions de régularité, toute fonction de la variété est évaluée par la méthode de quadrature empirique avec précision $\bar{\delta}$ quand $J \to \infty$. Nous présentons deux exemples nu-

E-mail addresses: patera@mit.edu (A.T. Patera), myano@utias.utoronto.ca (M. Yano).

https://doi.org/10.1016/j.crma.2017.10.020

mériques : une transformée inverse de Laplace et un traitement par base réduite d'une équation aux dérivées partielles non linéaire.

1. Introduction

In this note, we consider the integration of parameterized functions,

$$I(\mu) = \int_\Omega g(\mu; \xi)\, \mathrm{d}\xi \,, \tag{1}$$

for μ in the parameter domain $\mathcal{D} \subset \mathbb{R}^P$, $\Omega \subset \mathbb{R}^d$, and $g \in L^\infty(\mathcal{D}; L^\infty(\Omega))$. (We note that although Ω is parameter-independent, this spatial domain may be the result of a transformation from a parameter-dependent spatial domain through standard change-of-variable techniques.) Parameterized integrals arise in a variety of applications, from transform methods for ordinary differential equations, in which (say) ξ is frequency and μ includes time, to variational approximation of partial differential equations, in which ξ is a spatial coordinate and μ includes constitutive constants, sources, and geometric transformations.

We are interested in particular in the many-query context, in which $\mu \in \mathcal{D} \mapsto I(\mu)$ must be performed many times, often in real-time, for different values of μ in $\mathcal{D}$. We may thus gainfully consider offline–online approaches: an empirical quadrature rule — points and weights particularly optimized for (1) — is developed, once, in a relatively expensive offline stage; this efficient quadrature rule is then invoked, many times, in a very inexpensive online stage. The effort of the offline stage is justified, in fact amortized, over the many parameter queries $\mu \in \mathcal{D} \mapsto I(\mu)$ of the online stage.

One approach to (1) is interpolation-then-integration: we develop an interpolant for $g(\mu; \cdot)$ which then serves as a surrogate for $g(\mu; \xi)$ in (1); as an example of interpolation schemes for parametric functions, we cite the Empirical Interpolation Method [2]. Although interpolation-then-integration can be quite effective in practice, in fact the objectives and metrics associated with interpolation and integration are quite different, and thus a more direct approach — empirical quadrature rather than empirical interpolation — is also of interest.

An empirical quadrature procedure for parameterized functions is developed in [1] and further extended in [6]. These approaches consider an ℓ_2 framework and thus sparsity must be introduced explicitly, either through a heuristic sequential point selection process (as in [1]) or through an approximate ℓ_0 optimization (as in [6]); in both cases, a somewhat challenging non-negative least-squares problem must be addressed. In the current paper, we propose an ℓ_1 framework: a stronger norm which naturally yields sparse designs and which furthermore can be cast as a linear program (LP) efficiently treated by the dual simplex method. Our approach is an extension to the parametric context of the LP quadrature framework first proposed in [9] and further developed in [3].

2. Formulation

We define a parameter domain $\mathcal{D} \subset \mathbb{R}^P$, a point in which will be denoted $\mu = (\mu_1, \ldots, \mu_P)$, and an integration domain $\Omega \subset \mathbb{R}^d$, a point in which will be denoted $\xi = (\xi_1, \ldots, \xi_d)$. We then introduce a set of parameterized functions $g_m : \mathcal{D} \times \Omega \to \mathbb{R}$, $\forall m \in \mathbb{M}$; here $\mathbb{M} \equiv \{1, \ldots, M\}$ for M a finite positive integer. We shall assume that our set of functions satisfies a Lipschitz condition,

$$\sup_{m \in \mathbb{M}} \sup_{\mu', \mu'' \in \mathcal{D}^2} \| g_m(\mu'; \cdot) - g_m(\mu''; \cdot) \|_{L^\infty(\Omega)} \le L_g \| \mu' - \mu'' \|_2 \,, \tag{2}$$

for L_g a finite constant and $\|z\|_2$ the usual Euclidean norm (here) for $z \in \mathbb{R}^P$.

We next define the set of integrals of interest:

$$I_m(\mu) = \int_\Omega g_m(\mu; \xi)\, \mathrm{d}\xi \,, \forall m \in \mathbb{M}, \forall \mu \in \mathcal{D} \,. \tag{3}$$

We shall also require a "truth" quadrature,

$$I_m^{\mathrm{truth}}(\mu) = \sum_{i=1}^{\mathcal{N}} w_i^{\mathrm{truth}} g_m(\mu; \xi_i^{\mathrm{truth}}), \forall m \in \mathbb{M}, \forall \mu \in \mathcal{D} \,, \tag{4}$$

where $\{w_i^{\mathrm{truth}}\}_{i=1,\ldots,\mathcal{N}}$ and $\{\xi_i^{\mathrm{truth}}\}_{i=1,\ldots,\mathcal{N}}$ are the truth (non-negative) quadrature weights and truth quadrature points, respectively. We shall assume that (a) for ϵ a prescribed error tolerance,