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Probability theory

About the conditional value at risk of partial sums

Valeur à risque conditionnelle de sommes de variables aléatoires réelles

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ABSTRACT

In this note, we give normal approximation results for the conditional value at risk (CVaR) of partial sums of random variables satisfying moment assumptions. These results are based on Berry–Esseen-type bounds for transport costs in the central limit theorem and extensions of Cantelli's inequalities to the CVaR.

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RÉSUMÉ

Dans cette note, nous donnons des résultats d'approximation normale pour la CVaR d'une somme de variables aléatoires réelles satisfaisant des hypothèses de moments. Ces résultats sont fondés sur des bornes de type Berry–Esseen pour des coûts de transport dans le théorème limite central ainsi que sur des extensions des inégalités de Cantelli à la CVaR.

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1. Introduction

Let $(X_k)_{k>0}$ be a sequence of integrable and centered real-valued random variables. Set $S_n = X_1 + \cdots + X_n$ and $S_0 = 0$. In this note, we are interested in upper bounds on the conditional value at risk (CVaR) of S_n, as defined below.

Definition 1.1. Let X be a real-valued integrable random variable. The tail function H_X is defined by $H_X(t) = \mathbb{P}(X > t)$. In this note, the function Q_X is the cadlag inverse of H_X, which is defined by

$$Q_X(u) = \inf\{x : H_X(x) \le u\}.$$

We emphasize that $Q_X(u)$ is the value of the usual quantile function at point $1-u$. The function $\tilde{Q}_X$ is defined by

$$\tilde{Q}_X(u) = u^{-1} \int_0^u Q_X(s)\,ds \quad \text{for any } u \in]0,1]. \tag{1.1}$$

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With this definition, $\tilde{Q}_X$ is the conditional value at risk (CVaR) of X, also called "expected shortfall". It is worth noticing that $\tilde{Q}_X(u)$ is the expected value of X conditionally on the fact that X is larger than $Q_X(u)$.

Remark 1.1. The function $\tilde{Q}_X(u)$ is in fact the Hardy–Littlewood maximal function associated with the law of X, as defined by Hardy and Littlewood [6] in a well-known paper.

Throughout this note, we assume that $(S_n)_{n\geq 0}$ is a martingale in L^1. Our original motivation was to give deviation inequalities for $S_n^* = \max(S_0, S_1, \dots, S_n)$. Recall now that

$$Q_{S_n^*}(u) \leq \tilde{Q}_{S_n}(u) \text{ for any } u \in]0,1], \tag{1.2}$$

which shows that $Q_{S_n} \leq Q_{S_n^*} \leq \tilde{Q}_{S_n}$. This result may be found in [5]. From (1.2), any upper bound on the CVaR of S_n yields immediately the same upper bound on the quantile of S_n^*. Accordingly, from now on we will focus on conditional values at risk.

Let p be any real in $]1,\infty[$. For random variables X_k in L^p for some p in $]1,2]$, Rio [9, Theorem 4.1] gives the following upper bound on the conditional value at risk of S_n:

$$\tilde{Q}_{S_n}(1/z) \leq \|S_n\|_p \, z^{1/p}\big(1+(z-1)^{1-p}\big)^{-1/p} \text{ for any } z>1. \tag{1.3}$$

For example, when $p=2$, the above upper bound is equal to $\|S_n\|_2\sqrt{z-1}$. Our aim in this note is to improve (1.3) in the case $p\geq 2$. A possible way to improve this bound is to use global versions of the central limit theorem. Let $V_n = \operatorname{Var} S_n$. For independent and identically distributed square integrable random variables X_k, Agnew [1, Theorem 3.1] proved that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \big|H_{S_n}(t\sqrt{V_n}) - H_Y(t)\big| \,\mathrm{d}t = 0, \tag{1.4}$$

where Y is a random variable with standard normal law $N(0,1)$. From the elementary identity

$$\lim_{n\to\infty} \int_{\mathbb{R}} \big|H_{S_n}(t\sqrt{V_n}) - H_Y(t)\big| \mathrm{d}t = \int_0^1 \big|V_n^{-1/2} Q_{S_n}(u) - Q_Y(u)\big| \mathrm{d}u, \tag{1.5}$$

we immediately infer that

$$\big|V_n^{-1/2}\tilde{Q}_{S_n}(1/z) - \tilde{Q}_Y(1/z)\big| \leq z\int_{\mathbb{R}} \big|H_{S_n}(t\sqrt{V_n}) - H_Y(t)\big| \mathrm{d}t \tag{1.6}$$

for any $z>1$. It follows that $V_n^{-1/2}\tilde{Q}_{S_n}(1/z)$ converges to $\tilde{Q}_Y(1/z)$ as n tends to ∞. Since $\tilde{Q}_Y(1/z) \leq \sqrt{2\log z}$ (*see* Remark 3.1), (1.6) provides a much more efficient upper bound on $\tilde{Q}_{S_n}(1/z)$ than (1.3) for z large, at least for large values of n. However, the upper bound on the right-hand side in the above inequality is rapidly increasing as z tends to ∞. To improve (1.6), we will use Fréchet's L^r-distances or generalizations of these distances rather than the L^1-distance between the tail functions. For $r\geq 1$ and X and Y real-valued random variables in L^r with respective laws P_X and P_Y, the Fréchet distance W_r is defined by

$$W_r(P_X,P_Y) = \left(\int_0^1 |Q_X(u) - Q_Y(u)|^r \mathrm{d}u\right)^{1/r}. \tag{1.7}$$

As shown by Fréchet [4], $W_r(P_X,P_Y)$ is the minimal transport cost in L^r. In Section 2, we give upper bounds on $\tilde{Q}_X - \tilde{Q}_Y$ involving $W_r(P_X,P_Y)$ or generalizations of this distance. In Section 3, we apply the results of Section 2 to the conditional value at risk of sums of independent random variables in L^p for $p\geq 3$. Section 4 is devoted to stationary martingale differences sequences.

2. Conditional value at risk and power transport costs

Throughout this section, X and Y are real-valued random variables in L^r for some $r\geq 1$. Let ψ be a positive, measurable and integrable function on $]0,1[$, bounded from below by some positive constant. Let $L_{r,\psi}$ be the subspace of L^r of real-valued random variables Z such that

$$\int_0^1 |Q_Z(u)|^r \psi(u)\,\mathrm{d}u < \infty. \tag{2.1}$$

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