C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

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C. R. Acad. Sci. Paris, Ser. I

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Differential geometry

A note on the almost-one-half holomorphic pinching

Une note sur le pincement holomorphe presque un demi

Xiaodong Cao [1], Bo Yang [2,3]

Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, NY 14853-4201, USA

ARTICLE INFO

Article history:
Received 29 June 2017
Accepted after revision 29 September 2017
Available online xxxx

Presented by Jean-Pierre Demailly

ABSTRACT

Motivated by a previous work by Zheng and the second-named author, we study pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature. In particular, we prove a gap theorem following the work of Petersen and Tao on Riemannian manifolds with almost-quarter-pinched sectional curvature.

RÉSUMÉ

Motivés par un travail précédent de Zheng et du second auteur, nous étudions les constantes de pincement des variétés kählériennes compactes avec courbure sectionnelle holomorphe positive. En particulier, nous prouvons un théorème de l'écart s'appuyant sur le travail de Petersen et de Tao sur les variétés riemanniennes avec une courbure sectionnelle presque $\frac{1}{4}$-pincée.

1. The theorem

Let (M, J, g) be a complex manifold with a Kähler metric g, one can define the *holomorphic sectional curvature* (H) of any J-invariant real 2-plane $\pi = \mathrm{Span}\{X, JX\}$ by

$$H(\pi) = \frac{R(X, JX, JX, X)}{||X||^4}.$$

It is the Riemannian sectional curvature restricted on any J-invariant real 2-plane (p. 165, [18]). In terms of complex coordinates, it is equivalent to write

E-mail addresses: xiaodongcao@cornell.edu (X. Cao), boyang@math.cornell.edu, boyang@xmu.edu.cn (B. Yang).

[1] Research partially supported by a Simons Collaboration Grant.
[2] Research partially supported by an AMS-Simons Travel Grant.
[3] Current address: School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, 361005, China.

https://doi.org/10.1016/j.crma.2017.09.016
1631-073X/

$$H(\pi) = \frac{R(V, \overline{V}, V, \overline{V})}{||V||^4}$$

where $V = X - \sqrt{-1}JX \in T^{1,0}(M)$.

In this note, we study the pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature ($H > 0$). The goal is to prove the following rigidity result on a compact Kähler manifold with the almost-one-half pinching.

Theorem 1.1. *For any integer $n \geq 2$, there exists a positive constant $\epsilon(n)$ such that any compact Kähler manifold with $\frac{1}{2} - \epsilon(n) \leq H \leq 1$ of dimension n is biholomorphic to any of the following:*

(i) $\mathbb{CP}^n$,
(ii) $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$,
(iii) *an irreducible rank-2 compact Hermitian symmetric space of dimension n.*

Before we discuss the proof, let us review some background on compact Kähler manifolds with $H > 0$. The condition $H > 0$ is less understood and seems mysterious. For example, $H > 0$ does not imply positive Ricci curvature, though it leads to positive scalar curvature. Essentially, one has to work on a fourth-order tensor from the viewpoint of linear algebra, while usually the stronger notion of holomorphic bisectional curvature leads to bilinear forms.

Naturally, one may wonder if there is a characterization of such an interesting class of Kähler manifolds. In particular, Yau ([30] and [31]) asked if the positivity of holomorphic sectional curvature can be used to characterize the rationality of algebraic manifolds. For example, is such a manifold a rational variety? There is much progress on Kähler surfaces with $H > 0$. In 1975, Hitchin [17] proved that any compact Kähler surface with $H > 0$ must be a rational surface, and conversely he constructed examples of such metrics on any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^1})$. It remains an interesting question to find out if Kähler metrics with $H > 0$ exist on other rational surfaces.

In higher dimensions, much less is known on $H > 0$, except recent important works of Heier–Wong (see [16] for example). One of their results states that any projective manifold that admits a Kähler metric with $H > 0$ must be rationally connected. It could be possible that any compact Kähler manifold with $H > 0$ is in fact projective, again it is an open question.[4] We also remark that some generalization of Hitchin's construction of Kähler metrics of $H > 0$ in higher dimensions has been obtained in [2].

If Yau's conjecture is true, then how do we study complexities of rational varieties that admit Kähler metrics with $H > 0$? A naive thought is that the global and local holomorphic pinching constants of H should give a stratification among all such rational varieties. Here the local holomorphic pinching constant of a Kähler manifold (M, J, g) of $H > 0$ is the maximum of all $\lambda \in (0, 1]$ such that $0 < \lambda H(\pi') \leq H(\pi)$ for any J-invariant real 2-planes $\pi, \pi' \subset T_p(M)$ at any $p \in M$, while the global holomorphic pinching constant is the maximum of all $\lambda \in (0, 1]$ such that there exists a positive constant C so that $\lambda C \leq H(p, \pi) \leq C$ holds for any $p \in M$ and any J-invariant real 2-plane $\pi \subset T_p(M)$. Obviously, the global holomorphic pinching constant is no larger than the local one, and there are examples of Kähler metrics with different global and local holomorphic pinching constants on Hirzebruch manifolds ([29]).

In a previous work of Zheng and the second-named author [29], we observed the following result, which follows from some pinching equality on $H > 0$ due to Berger [3] and from recent works on nonnegative orthogonal bisectional curvature ([6], [9], [12], and [28]).

Proposition 1.2 *([29]). Let (M^n, g) be a compact Kähler manifold with $0 < \lambda \leq H \leq 1$ in the local sense for some constant λ, then the following holds:*

(1) *if $\lambda > \frac{1}{2}$, then M^n is bibolomorphic to $\mathbb{CP}^n$;*
(2) *if $\lambda = \frac{1}{2}$, then M^n satisfies one of the following*
  (i) *M^n is biholomorphic to $\mathbb{CP}^n$;*
  (ii) *M^n is holomorphically isometric to $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$ with a product metric of Fubini–Study metrics; moreover, each factor must have the same constant H;*
  (iii) *M^n is holomorphically isometric to an irreducible compact Hermitian symmeric space of rank 2 with its canonical Kähler–Einstein metric.*

Let us remark that in the case that Kähler manifold in Proposition 1.2 is projective and endowed with the induced metric from the Fubini–Study metric of the ambient projective space, a complete characterization of such a projective manifold and the corresponding embedding has been proved by Ros [25].

[4] After an earlier version of this preprint was submitted for consideration for publication in June 2017, X. Yang (arXiv:1708.06713) made important progress on these questions and proved that any compact Kähler manifold with $H > 0$ is projective and rationally connected.

Please cite this article in press as: X. Cao, B. Yang, A note on the almost-one-half holomorphic pinching, C. R. Acad. Sci. Paris, Ser. I (2017), https://doi.org/10.1016/j.crma.2017.09.016