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Simplicial complexes and closure systems induced by indistinguishability relations

Complexes simpliciaux et systèmes de clôture induits par les relations d'indistinguabilité

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, we develop in a more general mathematical context the notion of *indistinguishability*, which in graph theory has recently been investigated as a symmetry relation with respect to a fixed vertex subset. The starting point of our analysis is to consider a set Ω of functions defined on a universe set U and to define an equivalence relation $\equiv_A$ on U for any subset $A \subseteq \Omega$ in the following way: $u \equiv_A u'$ if $a(u) = a(u')$ for any function $a \in A$. By means of this family of relations, we introduce the *indistinguishability relation* $\approx$ on the power set $\mathcal{P}(\Omega)$ as follows: for $A, A' \in \mathcal{P}(\Omega)$, we set $A \approx A'$ if the relations $\equiv_A$ and $\equiv_{A'}$ coincide. We use then the indistinguishability relation $\approx$ to introduce several set families on Ω that have interesting order, matroidal and combinatorial properties. We call the above set families the *indistinguishability structures* of the *function system* (U, Ω). Furthermore, we obtain a closure system and an abstract simplicial complex interacting each other by means of three hypergraphs having relevance in both theoretical computer science and graph theory. The first part of this paper is devoted to investigate the basic mathematical properties of the indistinguishability structures for arbitrary function systems. The second part deals with some specific cases of study derived from simple undirected graphs and the usual Euclidean real line.

R É S U M É

Nous développons dans ce texte la notion d'*indistinguabilité* dans un contexte mathématique plus général. Cette notion a en effet été récemment étudiée en théorie des graphes, comme une relation de symétrie relativement aux sommets fixés. Le point de départ de notre analyse est de considérer un ensemble Ω de fonctions définies sur un ensemble univers U et de définir pour tout sous-ensemble $A \subset \Omega$ une relation d'équivalence $\equiv_A$ sur U par $u \equiv u'$ si $a(u) = a(u')$ pour toute fonction $a \in A$. Au moyen de cette famille de relations, nous introduisons la *relation d'indistinguabilité* $\approx$ sur l'ensemble puissance $\mathcal{P}(\Omega)$ de la façon suivante : pour $A, A' \in \mathcal{P}(\Omega)$, nous posons $A \approx A'$ si les relations $\equiv_A$ et $\equiv_{A'}$ coïncident. Nous utilisons cette relation d'indistinguabilité $\approx$ pour définir plusieurs familles d'ensembles sur Ω ayant d'intéressantes propriétés d'ordre, de matroïde et combinatoires.

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Nous appelons les familles d'ensembles ci-dessus les *structures indistinguables* du système de fonctions (U, Ω). De plus, nous obtenons un système de clôture et un complexe simplicial abstrait interagissant l'un l'autre au travers de trois hypergraphes, qui sont significatifs aussi bien en théorie des graphes qu'en informatique théorique. La première partie du texte est dédiée à l'étude les propriétés mathématiques élémentaires des structures d'indistinguabilité pour les systèmes de fonctions arbitraires. La seconde partie traite de quelques cas particuliers dérivés des graphes non orientés simples et de la droite euclidienne réelle usuelle.

1. Introduction

In graph theory, the symmetry notion is classically related to the properties of the automorphism group $Aut(G)$ of a graph G (see [34]), and it is also object of deep algebraic investigations (see [39]). More in general, since a graph can be considered as a particular type of metric space, the symmetry notion in this context has been investigated in [40,46,47,53]. Recently, in [26,29] a type of notion of symmetry for simple undirected graphs has been investigated. More in detail, when we have a simple undirected graph G and when we fix a vertex subset A of G, we can consider the following equivalence relation $\equiv_A$ (called *A-symmetry*) on the vertex set $V(G)$. For $v, v' \in V(G)$, we set

$$v \equiv_A v' :\Longleftrightarrow N_G(v) \cap A = N_G(v') \cap A,$$

where $N_G(v)$ is the usual open neighborhood of a vertex v of G. In [26,29], some new hypergraph families induced by the above A-symmetry have been introduced and studied in relation to motivations derived from database theory and related fields (see [26] for details). Based on some results obtained in [29], in [30] it has been introduced a new binary operation $\circ$ on a vertex subset family of G whose automorphism group (with respect to $\circ$) is isomorphic to a subgroup of $Aut(G)$ (for other works on similar topics, see also [16,17]).

On the other hand, there are actually many researches and studies in discrete mathematics whose main theoretical results are strictly related to motivations derived from computer science and affine disciplines. For example, in [43–45] several types of results concerning the Cayley graphs, graph algebras and directed graphs are connected with automata theory and data mining methods. In [41,52,55], matroids have been used in their connections with new topics in theoretical computer science. From another perspective, in [2–5,36] some classes of bipartite graphs have been studied with FCA methods [35]. Again, in [20–22] some types of homotopy relations for graphs that are strictly related with database theory have been introduced. In [12,13,15,19,25], some classes of lattices of signed integer partitions [1,42] have been investigated as sequential and discrete dynamical systems [50,51], with motivations related to some extremal combinatorial sum problems [23,24].

In this paper, we develop the above notion of A-symmetry (that in the study of data tables is usually called *indiscernibility relation*) in a purely mathematical context. To be more specific, let Ω be an arbitrary non-empty set. Usually, in mathematics, the elements of Ω do not have a well-specified nature. The basic idea of this paper is to assume that the elements of Ω are *functions*, whose domain is some *point* set U and whose codomain is some *value* set Λ. We say therefore that a *function system* is a triple $\mathcal{I} = \langle U, \Omega, \Lambda\rangle$, where U is a set of *points* $u, v, z, \ldots$ and Ω is a set of *functions* $a, b, c, \ldots$ that have domain U and codomain Λ. Let us note that a function system is a type of very general structure, which can be found in many different mathematical contexts, both in the finite and in the infinite cases.
We can uniquely associate with the function system $\mathcal{I}$ the *global map* $F : U \times \Omega \to \Lambda$ defined by

$$F(u, a) := a(u),$$

for any $u \in U$ and $a \in \Omega$. Let then $\mathcal{I} = \langle U, \Omega, F, \Lambda\rangle$ be a given function system with global map F.

If $A \subseteq \Omega$ and $u, u' \in U$, we set

$$u \equiv_A u' :\Longleftrightarrow a(u) = a(u') \qquad \forall a \in A.$$

In an approximation geometry for conceptual patterns [48], the equivalence relation $\equiv_A$ is called *A-indiscernibility relation* and it is used when the functions in Ω are substituted by specific *attributes* (or, equivalently, *properties*) of a set of *objects*. In this paper, we continue to use this terminology, and we call the set partition of U induced $\equiv_A$ the *A-indiscernibility partition* and denote it by $\pi(A)$.

We consider now the following equivalence relation $\approx$ defined on the power set $\mathcal{P}(\Omega)$ of Ω. If A and A' are any two subsets of Ω, we set

$$A \approx A' :\Longleftrightarrow \pi(A) = \pi(A'),$$

and we call $\approx$ the *indistinguishability relation* on $\mathcal{I}$. If $A \subseteq \Omega$, we denote by $[A]_\approx$ the equivalence class of A with respect to the equivalence relation $\approx$ and we call $[A]_\approx$ the *indistinguishability class* of A. In this paper, we will show that the indistinguishability relation induces a very rich mathematical structure on the power set $\mathcal{P}(\Omega)$, and the richness of this