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A variational principle in the parametric geometry of numbers, with applications to metric Diophantine approximation

Un principe variationnel en géométrie paramétrique des nombres, illustré par des applications à la théorie métrique de l'approximation diophantienne

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ABSTRACT

We establish a new connection between metric Diophantine approximation and the parametric geometry of numbers by proving a variational principle facilitating the computation of the Hausdorff and packing dimensions of many sets of interest in Diophantine approximation. In particular, we show that the Hausdorff and packing dimensions of the set of singular $m \times n$ matrices are both equal to $mn\left(1 - \frac{1}{m+n}\right)$, thus proving a conjecture of Kadyrov, Kleinbock, Lindenstrauss, and Margulis as well as answering a question of Bugeaud, Cheung, and Chevallier. Other applications include computing the dimensions of the sets of points witnessing conjectures of Starkov and Schmidt.

RÉSUMÉ

Nous établissons un nouveau lien entre la théorie métrique de l'approximation diophantienne et la géométrie paramétrique des nombres, en démontrant un principe variationnel permettant le calcul des dimensions de Hausdorff et d'entassement de nombreux ensembles d'intérêt en approximation diophantienne. Comme cas particulier, nous démontrons que les dimensions de Hausdorff et d'entassement de l'ensemble des matrices singulières de dimensions $m \times n$ sont toutes deux égales à $mn\left(1 - \frac{1}{m+n}\right)$, démontrant ainsi une conjecture de Kadyrov, Kleinbock, Lindenstrauss et Margulis, et répondant par là même à une question soulevée par Bugeaud, Cheung et Chevallier. D'autres exemples d'applica-

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tion incluent le calcul des dimensions des ensembles de points satisfaisant des conjectures énoncées par Starkov et Schmidt.

1. Main results

The notion of singularity (in the sense of Diophantine approximation) was introduced by Khintchine, first in 1937 in the setting of simultaneous approximation [11], and later in 1948 in the more general setting of matrix approximation [12].[1] Since then this notion has been studied within Diophantine approximation and allied fields, see Moshchevitin's 2010 survey [13]. An $m \times n$ matrix A is called *singular* if for all $\epsilon > 0$, there exists Q_ϵ such that for all $Q \geq Q_\epsilon$, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq \epsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| \leq Q\,.$$

Here $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^m$ or $\mathbb{R}^n$. We denote the set of singular $m \times n$ matrices by $\mathrm{Sing}(m, n)$. For 1×1 matrices (i.e. numbers), being singular is equivalent to being rational, and in general any matrix A which satisfies an equation of the form $A\mathbf{q} = \mathbf{p}$, with $\mathbf{p}, \mathbf{q}$ integral and $\mathbf{q}$ nonzero, is singular. However, Khintchine proved that there exist singular 2×1 matrices whose entries are linearly independent over $\mathbb{Q}$ [10, Satz II], and his argument generalizes to the setting of $m \times n$ matrices for all $(m, n) \neq (1, 1)$. The name *singular* derives from the fact that $\mathrm{Sing}(m, n)$ is a Lebesgue nullset for all m, n, see e.g. [11, p. 431] or [2, Chapter 5, §7]. Note that singularity is a strengthening of the property of *Dirichlet improvability* introduced by Davenport and Schmidt [6].

In contrast to the measure zero result mentioned above, the computation of the Hausdorff dimension of $\mathrm{Sing}(m, n)$ has been a challenge that so far only met with partial progress. The first breakthrough was made in 2011 by Cheung [3], who proved that the Hausdorff dimension of $\mathrm{Sing}(2, 1)$ is $4/3$; this was extended in 2016 by Cheung and Chevallier [4], who proved that the Hausdorff dimension of $\mathrm{Sing}(m, 1)$ is $m^2/(m + 1)$ for all $m \geq 2$; while most recently Kadyrov, Kleinbock, Lindenstrauss, and Margulis [8] proved that the Hausdorff dimension of $\mathrm{Sing}(m, n)$ is at most $\delta_{m,n} := mn\big(1 - \frac{1}{m+n}\big)$, and went on to conjecture that their upper bound is sharp for all $(m, n) \neq (1, 1)$ (see also [1, Problem 1]).

In this paper, we announce a proof that their conjecture is correct. We will also show that the packing dimension of $\mathrm{Sing}(m, n)$ is the same as its Hausdorff dimension, thus answering a question of Bugeaud, Cheung, and Chevallier [1, Problem 7]. To summarize:

Theorem 1.1. *For all $(m, n) \neq (1, 1)$, we have*

$$\dim_H(\mathrm{Sing}(m, n)) = \dim_P(\mathrm{Sing}(m, n)) = \delta_{m,n} \stackrel{\text{def}}{=} mn\big(1 - \tfrac{1}{m+n}\big),$$

where $\dim_H(S)$ and $\dim_P(S)$ denote the Hausdorff and packing dimensions of a set S, respectively.

1.1. Dani correspondence

The set of singular matrices is linked to homogeneous dynamics via the *Dani correspondence principle*. For each $t \in \mathbb{R}$ and for each matrix A, let

$$g_t \stackrel{\text{def}}{=} \begin{bmatrix} e^{t/m} I_m & \\ & e^{-t/n} I_n \end{bmatrix}, \qquad u_A \stackrel{\text{def}}{=} \begin{bmatrix} I_m & A \\ & I_n \end{bmatrix},$$

where I_k denotes the k-dimensional identity matrix. Finally, let $d = m+n$, and for each $j = 1, \ldots, d$, let $\lambda_j(\Lambda)$ denote the jth successive minimum of a lattice $\Lambda \subset \mathbb{R}^d$ (with respect to some fixed norm on $\mathbb{R}^d$), i.e. the infimum of λ such that the set $\{\mathbf{r} \in \Lambda : \|\mathbf{r}\| \leq \lambda\}$ contains j linearly independent vectors. Then the Dani correspondence principle is a dictionary between the Diophantine properties of a matrix A on the one hand, and the dynamical properties of the orbit $(g_t u_A \mathbb{Z}^d)_{t \geq 0}$ on the other. A particular example is the following result:

Theorem 1.2 *([5, Theorem 2.14]). An $m \times n$ matrix A is singular if and only if the trajectory $(g_t u_A \mathbb{Z}^d)_{t\geq 0}$ is divergent in the space of unimodular lattices in $\mathbb{R}^d$, or equivalently if*

$$\lim_{t\to\infty} \lambda_1(g_t u_A \mathbb{Z}^d) = 0.$$

[1] Although Khintchine's 1926 paper [10] includes a proof of the existence of 2×1 and 1×2 matrices possessing a certain property which implies that they are singular, it does not include a definition of singularity nor discuss any property equivalent to singularity.