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A proof of energy gap for Yang–Mills connections

Une preuve du gap d'énergie pour les connexions de Yang–Mills

Teng Huang

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

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ABSTRACT

In this note, we prove an $L^{\frac{n}{2}}$-energy gap result for Yang–Mills connections on a principal G-bundle over a compact manifold without using the Lojasiewicz–Simon gradient inequality ([2] Theorem 1.1).

RÉSUMÉ

Dans cette note, nous démontrons un résultat concernant le gap d'énergie $L^{\frac{n}{2}}$ pour les connexions de Yang–Mills sur un fibré principal de groupe structural G sur une variété compacte, sans utiliser l'inégalité du gradient de Lojasiewicz–Simon.

1. Introduction

Let X be a compact n-dimensional Riemannian manifold endowed with a smooth Riemannian metric g, $P \to X$ a principal G-bundle over X, where G is a compact Lie group. We define the Yang–Mills functional by

$$YM(A) = \int_X |F_A|^2 \mathrm{d}vol_g,$$

where A is a C^∞-connection on P and F_A is the curvature of A.

A connection A on P is called a Yang–Mills connection if it is a critical point of YM, i.e. it obeys the Yang–Mills equation with respect to the metric g:

$$d_A^* F_A = 0. \tag{1.1}$$

In [2], Feehan proved an $L^{\frac{n}{2}}$-energy gap result for Yang–Mills connections on the principal G-bundle P over an arbitrary closed smooth Riemannian manifold with dimension $n \geq 2$ ([2] Theorem 1.1). Feehan applied the Lojasiewicz–Simon gradient

E-mail address: oula143@mail.ustc.edu.cn.

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inequality ([2] Theorem 3.2) to remove a positivity hypothesis on the Riemannian curvature tensors in a previous $L^{\frac{n}{2}}$-energy gap result due to Gerhardt [3] (Theorem 1.2).

In this note, we give another proof of this $L^{\frac{n}{2}}$-energy gap result of Yang–Mills connection without using the Lojasiewicz–Simon gradient inequality.

Theorem 1.1. *([2] Theorem 1.1) Let X be a compact Riemannian manifold without boundary of dimension $n \geq 2$ endowed with a smooth Riemannian metric g, P be a G-bundle over X. Then, either any smooth Yang–Mills connection A over X with compact Lie group G satisfies*

$$\int_X |F_A|^{\frac{n}{2}} \mathrm{d}vol_g \geq C_0$$

for a constant $C_0 > 0$ depending only on X, n, G, or the connection A is flat.

2. Preliminaries and basic estimates

We shall generally adhere to the now standard gauge-theory conventions and notation of Donaldson and Kronheimer [1] and Feehan [2]. Throughout our article, G denotes a compact Lie group and P a smooth principal G-bundle over a compact Riemannian manifold X of dimension $n \geq 2$ endowed with a Riemannian metric g, $\mathfrak{g}_P$ denote the adjoint bundle of P, endowed with a G-invariant inner product and $\Omega^p(X, \mathfrak{g}_P)$ denote the smooth p-forms with values in $\mathfrak{g}_P$. Given a connection on P, we denote by ∇_A the corresponding covariant derivative on $\Omega^*(X, \mathfrak{g}_P)$ induced by A and the Levi-Civita connection of X. Let d_A denote the exterior derivative associated with ∇_A.

For $u \in L^p(X, \mathfrak{g}_P)$, where $1 \leq p < \infty$ and k is an integer, we denote

$$\|u\|_{L^p_{k,A}(X)} := \Big(\sum_{j=0}^{k} \int_X |\nabla_A^j u|^p \mathrm{d}vol_g\Big)^{1/p},$$

where $\nabla_A^j := \nabla_A \circ \ldots \circ \nabla_A$ (repeated j times for $j \geq 0$). For $p = \infty$, we denote

$$\|u\|_{L^\infty_{k,A}(X)} := \sum_{j=0}^{k} ess \sup_X |\nabla_A^j u|.$$

At first, we review a key result due to Uhlenbeck for the connections with L^p-small curvature $(2p > n)$ [5], which provides the existence of a flat connection Γ on P, of a global gauge transformation u of A to Coulomb gauge with respect to Γ, and of a Sobolev norm estimate for the distance between Γ and A.

Theorem 2.1. *([5] Corollary 4.3 and [2] Theorem 5.1) Let X be a closed, smooth manifold of dimension $n \geq 2$ endowed with a Riemannian metric, g, and G be a compact Lie group, and $2p > n$. Then there are constants, $\varepsilon = \varepsilon(n, g, G, p) \in (0, 1]$ and $C = C(n, g, G, p) \in [1, \infty)$, with the following property. Let A be a L^p_1 connection on a principal G-bundle P over X. If the curvature F_A obeys*

$$\|F_A\|_{L^p(X)} \leq \varepsilon,$$

then there exist a flat connection, $|\Gamma|$, on P, and a gauge transformation $u \in L^p_2(X)$ such that

(1) $d^*_\Gamma\big(u^*(A) - \Gamma\big) = 0$ *on* X,
(2) $\|u^*(A) - \Gamma\|_{L^p_{1,\Gamma}} \leq C\|F_A\|_{L^p(X)}$ *and*
(3) $\|u^*(A) - \Gamma\|_{L^{\frac{n}{2}}_{1,\Gamma}} \leq C\|F_A\|_{L^{\frac{n}{2}}(X)}$.

Next, we also review another key result due to Uhlenbeck concerning an a priori estimate for the curvature of a Yang–Mills connection over a closed Riemannian manifold.

Theorem 2.2. *([4] Theorem 3.5 and [2] Corollary 4.6) Let X be a compact manifold of dimension $n \geq 2$ endowed with a Riemannian metric g, let A be a smooth Yang–Mills connection with respect to the metric g on a smooth G-bundle P over X. Then there exist constants $\varepsilon = \varepsilon(X, n, g) > 0$ and $C = C(X, n, g)$ with the following property. If the curvature F_A obeys*

$$\|F_A\|_{L^{\frac{n}{2}}(X)} \leq \varepsilon,$$

then

$$\|F_A\|_{L^\infty(X)} \leq C\|F_A\|_{L^2(X)}.$$