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Partial differential equations/Theory of signals

Rigidity of optimal bases for signal spaces

Rigidité des bases optimales pour les espaces de signaux

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ABSTRACT

We discuss optimal L^2-approximations of functions controlled in the H^1-norm. We prove that the basis of eigenfunctions of the Laplace operator with Dirichlet boundary condition is the only orthonormal basis (b_i) of L^2 that provides an optimal approximation in the sense of

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \geq 1.$$

This solves an open problem raised by Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel, and N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

RÉSUMÉ

On s'intéresse à l'approximation optimale pour la norme L^2 de fonctions contrôlées en norme H^1. On prouve que la base des fonctions propres du laplacien avec condition de Dirichlet au bord est l'unique base orthonormale (b_i) de L^2 qui réalise une approximation optimale au sens de

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \geq 1.$$

Ceci résout un problème ouvert posé par Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel et N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

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1. Introduction and main result

This note is a follow-up of the papers by Y. Aflalo, H. Brezis and R. Kimmel [2] and Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel and N. Sochen [1].

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $e = (e_i)$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ is the ordered sequence of eigenvalues repeated according to their multiplicity.

We first recall a very standard result:

Theorem 1.1. *We have, for all $n \geq 1$,*

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 \leq \frac{\| \nabla f \|_{L^2}^2}{\lambda_{n+1}} \qquad \forall f \in H_0^1(\Omega). \tag{2}$$

Here and throughout the rest of this paper $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.

Indeed, we may write

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 = \left\| \sum_{i=n+1}^{+\infty} (f, e_i) e_i \right\|_{L^2}^2 = \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{3}$$

On the other hand,

$$\|\nabla f\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i (f, e_i)^2 \geq \sum_{i=n+1}^{+\infty} \lambda_i (f, e_i)^2 \geq \lambda_{n+1} \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{4}$$

Combining (3) and (4) yields (2). □

The authors of [2] and [1] have investigated the "optimality" in various directions of the basis (e_i), with respect to inequality (2). Here is one of their results restated in a slightly more general form.

Theorem 1.2 *(Theorem 3.1 in [2]). There is no integer $n \geq 1$, no constant $0 \leq \alpha < 1$ and no sequence $(\psi_i)_{1 \leq i \leq n}$ in $L^2(\Omega)$ such that*

$$\left\| f - \sum_{i=1}^{n} (f, \psi_i) \psi_i \right\|_{L^2}^2 \leq \frac{\alpha}{\lambda_{n+1}} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5}$$

The proof in [2] relies on the Fischer–Courant max–min principle (see Remark 3.3 below). For the convenience of the reader, we present a very elementary proof based on a simple and efficient device originally due to H. Poincaré [5, pp. 249–250] (and later rediscovered by many people, e.g., H. Weyl [7, p. 445] and R. Courant [3, pp. 17–18]; see also H. Weinberger [6, p. 56] and P. Lax [4, p. 319]).

Suppose not, and set

$$f = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n + c_{n+1} e_{n+1} \tag{6}$$

where $c = (c_1, c_2, \cdots, c_n, c_{n+1}) \in \mathbb{R}^{n+1}$. The under-determined linear system

$$(f, \psi_i) = 0, \qquad \forall i = 1, \cdots, n \tag{7}$$

of n equations with $n+1$ unknowns admits a non-trivial solution. Inserting f into (5) yields

$$\lambda_{n+1} \sum_{i=1}^{n+1} c_i^2 \leq \alpha \sum_{i=1}^{n+1} \lambda_i c_i^2 \leq \alpha \lambda_{n+1} \sum_{i=1}^{n+1} c_i^2. \tag{8}$$

Therefore $\sum_{i=1}^{n+1} c_i^2 = 0$ and thus $c = 0$. A contradiction. This proves Theorem 1.2. □

The authors of [1] were thus led to investigate the question of whether inequality (2) holds *only* for the orthonormal bases consisting of eigenfunctions corresponding to ordered eigenvalues. They established that a "discrete", i.e., finite-dimensional, version does hold; see [1, Theorem 2.1] and Remark 3.2 below. But their proof of "uniqueness" could not be adapted to the infinite-dimensional case (because it relied on a "descending" induction). It was raised there as an open problem (see [1, p. 1166]). Our next result solves this problem.