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Kimura-finiteness of quadric fibrations over smooth curves

Finitude à la Kimura de fibrations en quadriques sur des courbes lisses

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A B S T R A C T

Making use of the recent theory of noncommutative mixed motives, we prove that the Voevodsky's mixed motive of a quadric fibration over a smooth curve is Kimura-finite.

R É S U M É

Utilisant la théorie récente des motifs non commutatifs, nous prouvons que le motif mixte de Voevodsky d'une fibration en quadriques sur une courbe lisse est fini au sens de Kimura.

1. Introduction

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal category. Given a partition λ of an integer $n \geq 1$, consider the corresponding irreducible $\mathbb{Q}$-linear representation V_λ of the symmetric group $\mathfrak{S}_n$ and the associated idempotent $e_\lambda \in \mathbb{Q}[\mathfrak{S}_n]$. Under these notations, the Schur-functor $S_\lambda : \mathcal{C} \to \mathcal{C}$ sends an object a to the direct summand of $a^{\otimes n}$ determined by e_λ. In the particular case of the partition $\lambda = (1, \ldots, 1)$, resp. $\lambda = (n)$, the associated Schur-functor $\wedge^n := S_{(1,\ldots,1)}$, resp. $\mathrm{Sym}^n := S_{(n)}$, is called the *nth wedge product*, resp. the *nth symmetric product*. Following Kimura [11], an object $a \in \mathcal{C}$ is called *even-dimensional*, resp. *odd-dimensional*, if $\wedge^n(a)$, resp. $\mathrm{Sym}^n(a) = 0$, for some $n \gg 0$. The biggest integer $\mathrm{kim}_+(a)$, resp. $\mathrm{kim}_-(a)$, for which $\wedge^{\mathrm{kim}_+(a)} \neq 0$, resp. $\mathrm{Sym}^{\mathrm{kim}_-(a)}(a) \neq 0$, is called the *even*, resp. *odd*, *Kimura-dimension of* a. An object $a \in \mathcal{C}$ is called *Kimura-finite* if $a \simeq a_+ \oplus a_-$, with a_+ even-dimensional and a_- odd-dimensional. The integer $\mathrm{kim}(a) = \mathrm{kim}_+(a_+) + \mathrm{kim}_-(a_-)$ is called the *Kimura-dimension of* a.

Voevodsky introduced in [20] an important triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (over a perfect base field k). By construction, this category is $\mathbb{Q}$-linear, idempotent complete, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor $M(-)_{\mathbb{Q}} : \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, defined on smooth k-schemes. An important

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open problem[2] is the classification of all the Kimura-finite mixed motives and the computation of the corresponding Kimura-dimensions. On the negative side, O'Sullivan constructed a certain smooth surface S whose mixed motive $M(S)_{\mathbb{Q}}$ is *not* Kimura-finite; consult [14, §5.1] for details. On the positive side, Guletskii [7] and Mazza [14] proved, independently, that the mixed motive $M(C)_{\mathbb{Q}}$ of every smooth curve C is Kimura-finite.

The following result bootstraps Kimura-finiteness from smooth curves to quadric fibrations.

Theorem 1.1. *Let k be a field, C a smooth k-curve, and $q\colon Q \to C$ a flat quadric fibration of relative dimension $d-2$. Assume that Q is smooth and that q has only* simple degenerations, *i.e. that all the fibers of q have corank ≤ 1. Under these assumptions, the following holds:*

(i) *when d is even, the mixed motive $M(Q)_{\mathbb{Q}}$ is Kimura-finite. Moreover, we have the following equality, $\mathrm{kim}(M(Q)_{\mathbb{Q}}) = \mathrm{kim}(M(\widetilde{C})_{\mathbb{Q}}) + (d-2)\mathrm{kim}(M(C)_{\mathbb{Q}})$, where $D \hookrightarrow C$ stands for the finite set of critical values of q and $\widetilde{C}$ for the discriminant double cover of C (ramified over D);*
(ii) *when d is odd, k is algebraically closed, and $1/2 \in k$, the mixed motive $M(Q)_{\mathbb{Q}}$ is Kimura-finite. Moreover, we have the following equality $\mathrm{kim}(M(Q)_{\mathbb{Q}}) = \#D + (d-1)\mathrm{kim}(M(C)_{\mathbb{Q}})$.*

To the best of the author's knowledge, Theorem 1.1 is new in the literature. It not only provides new examples of Kimura-finite mixed motives, but also computes the corresponding Kimura dimensions.

Remark 1. In the particular case where k is algebraically closed and Q, C are moreover projective, Vial proved in [19, Cor. 4.4] that the Chow motive $\mathfrak{h}(Q)_{\mathbb{Q}}$ is Kimura-finite. Since the category of Chow motives embeds fully-faithfully into $\mathrm{DM_{gm}}(k)_{\mathbb{Q}}$ (see [20, §4]), we then obtain in this particular case an alternative "geometric" proof of the Kimura-finiteness of $M(Q)_{\mathbb{Q}}$. Moreover, when $k = \mathbb{C}$ and d is odd, Bouali refined Vial's work by showing that $\mathfrak{h}(Q)_{\mathbb{Q}} \simeq \mathbb{Q}(-\frac{d-1}{2})^{\oplus \# D} \oplus \bigoplus_{i=0}^{d-2} \mathfrak{h}(C)_{\mathbb{Q}}(-i)$; see [4, Rk. 1.10(i)]. In this particular case, this leads to an alternative "geometric" computation of the Kimura-dimension of $M(Q)_{\mathbb{Q}}$.

2. Preliminaries

Throughout the article, k denotes a base field of arbitrary characteristic.

Dg categories. For a survey on dg categories, consult Keller's ICM talk [9]. In what follows, we write $\mathrm{dgcat}(k)$ for the category of (essentially small) dg categories and dg functors. Every (dg) k-algebra gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks, since the category of perfect complexes $\mathrm{perf}(X)$ of every k-scheme X (or, more generally, algebraic stack $\mathcal{X}$) admits a canonical dg enhancement $\mathrm{perf_{dg}}(X)$; consult [9, §4.6][13] for details.

Noncommutative mixed motives. For a book, resp. survey, on noncommutative motives, consult [15], resp. [16]. Recall from [15, §8.5.1] the construction of Kontsevich's triangulated category of noncommutative mixed motives $\mathrm{NMot}(k)$; denoted by $\mathrm{NMot}^{\mathbb{A}^1}_{\mathrm{loc}}(k)$ in *loc. cit.* By construction, this category is idempotent complete, closed symmetric monoidal, and comes equipped with a symmetric monoidal functor $U\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$. In what follows, given a k-scheme X, we write $U(X)$ instead of $U(\mathrm{perf_{dg}}(X))$.

Root stacks. Let X be a k-scheme, $\mathcal{L}$ a line bundle on X, $\sigma \in \Gamma(X, \mathcal{L})$ a global section, and $r > 0$ an integer. In what follows, we write $D \hookrightarrow X$ for the zero locus of σ. Recall from [5, Def. 2.2.1] (see also [1, Appendix B]) that the associated *root stack* is defined as the following fiber-product of algebraic stacks

$$\begin{tikzcd} \sqrt[r]{(\mathcal{L},\sigma)/X} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\theta_r"] \\ X \arrow[r, "(\mathcal{L},\sigma)"'] & {[\mathbb{A}^1/\mathbb{G}_m]}, \end{tikzcd}$$

where θ_r stands for the morphism induced by the rth power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$.

Proposition 2.1. *We have $U(\sqrt[r]{(\mathcal{L},\sigma)/X}) \simeq U(D)^{\oplus(r-1)} \oplus U(X)$ whenever X and D are k-smooth.*

[2] Among other consequences, Kimura-finiteness implies rationality of the motivic zeta function.