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Partial differential equations

Observation estimate for kinetic transport equations by diffusion approximation

Inégalité d'observation pour des équations cinétiques linéaires par l'approximation de diffusion

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A R T I C L E I N F O

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A B S T R A C T

We study the unique continuation property for the neutron transport equation and for a simplified model of the Fokker–Planck equation in a bounded domain with absorbing boundary condition. An observation estimate is derived. It depends on the smallness of the mean free path and the frequency of the velocity average of the initial data. The proof relies on the well-known diffusion approximation under convenience scaling and on the basic properties of this diffusion. Eventually, we propose a direct proof for the observation at one time of parabolic equations. It is based on the analysis of the heat kernel.

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R É S U M É

L'objet de cet article est l'observation (et aussi la continuation unique) pour des solutions d'équations cinétiques linéaires avec, comme opérateur de collision, soit un modèle simplifié de l'équation de la neutronique, soit un opérateur de Fokker–Planck linéarisé. À l'aide de l'approximation de la diffusion, une inégalité d'observation en un temps donné est obtenue. Elle dépend du libre parcours moyen (ou de l'opacité du milieu) et de la fréquence de la moyenne de la donnée initiale. En plus de l'approximation de la diffusion, on utilise l'inégalité d'observation en temps fixé pour la diffusion. Pour cette dernière, on propose une nouvelle démonstration directe avec des estimations à poids utilisant la paramétrix à l'ordre zéro du noyau de la chaleur.

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1. Introduction

This article is devoted to the question of unique continuation for linear kinetic transport equation with a scattering operator in the diffusive limit. Let Ω be a bounded open subset of $\mathbb{R}^d$, $d > 1$, with boundary $\partial\Omega$ of class C^2. Consider in $\{(x, v) \in \Omega \times \mathbb{S}^{d-1}\} \times \mathbb{R}_t^+$ the transport equation in the v direction with a scattering operator S and the absorbing boundary condition

$$\begin{cases} \partial_t f + \dfrac{1}{\epsilon} v \cdot \nabla f + \dfrac{a}{\epsilon^2} S(f) = 0 & \text{in } \Omega \times \mathbb{S}^{d-1} \times (0, +\infty) , \\ f = 0 & \text{on } \left(\partial\Omega \times \mathbb{S}^{d-1}\right)_- \times (0, +\infty) , \\ f(\cdot, \cdot, 0) = f_0 \in L^2(\Omega \times \mathbb{S}^{d-1}) , \end{cases} \tag{1.1}$$

where $\epsilon \in (0, 1]$ is a small parameter and $a \in L^\infty(\Omega)$ is a scattering opacity satisfying $0 < c_{\min} \leq a(x) \leq c_{\max} < \infty$. Here, $\nabla = \nabla_x$ and $\left(\partial\Omega \times \mathbb{S}^{d-1}\right)_- = \left\{(x, v) \in \partial\Omega \times \mathbb{S}^{d-1}; v \cdot \vec{n}_x < 0\right\}$ where $\vec{n}_x$ is the unit outward normal field at $x \in \partial\Omega$.
Two standard examples of scattering operators $S : f \mapsto Sf$ are given below:

- the neutron scattering operator,

$$Sf = f - \langle f \rangle \text{ where } \langle f \rangle (x, t) = \frac{1}{\left|\mathbb{S}^{d-1}\right|} \int_{\mathbb{S}^{d-1}} f(x, v, t) \, dv;$$

- the Fokker–Planck scattering operator,

$$Sf = -\frac{1}{d-1} \Delta_{\mathbb{S}^{d-1}} f, \text{ where } \Delta_{\mathbb{S}^{d-1}} \text{ is the Laplace–Beltrami operator on } \mathbb{S}^{d-1}.$$

Recall that such operators have the properties of self-adjointness and $Sv = v$, which imply that $\langle v \cdot \nabla Sf \rangle = \langle v \cdot \nabla f \rangle$.

Let ω be a nonempty open subset of Ω. Suppose that we observe the solution f at time $T > 0$ and on ω, i.e. $f(x, v, T)_{|(x,v)\in\omega\times\mathbb{S}^{d-1}}$ is known. A classical inverse problem consists in recovering at least one solution, and in particular its initial data, which fits the observation on $\omega \times \mathbb{S}^{d-1} \times \{T\}$. Our problem of unique continuation is: with how many initial data will the corresponding solution achieve the given observation $f(x, v, T)_{|(x,v)\in\omega\times\mathbb{S}^{d-1}}$? Here ϵ is a small parameter and it is natural to focus on the limit solution. This is the diffusion approximation saying that the solution f converges to a solution to a parabolic equation when ϵ tends to 0 (see [3,8,16,2,5,6,4]). In this framework, two remarks can be made:

- for our scattering operator, there holds

$$\|f - \langle f \rangle\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)} \leq \epsilon \frac{1}{\sqrt{2c_{\min}}} \|f_0\|_{L^2(\Omega\times\mathbb{S}^{d-1})}.$$

  For the operator of neutron transport, one uses a standard energy method by multiplying both sides of the first line of (1.1) by f and integrating over $\Omega \times \mathbb{S}^{d-1} \times (0, T)$. For the Fokker–Planck scattering operator, one combines the standard energy method as above and Poincaré inequality

$$\|f - \langle f \rangle\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)} \leq \frac{1}{\sqrt{d-1}} \left\|\nabla_{\mathbb{S}^{d-1}} f\right\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)} \leq \epsilon \frac{1}{\sqrt{2c_{\min}}} \|f_0\|_{L^2(\Omega\times\mathbb{S}^{d-1})}.$$

- In the sense of distributions in Ω, for any $t \geq 0$, the average of f solves the following parabolic equation

$$\partial_t \langle f \rangle - \frac{1}{d} \nabla \cdot \left(\frac{1}{a} \nabla \langle f \rangle\right) = \nabla \cdot \left(\frac{1}{a} \langle (v \otimes v) \nabla (f - \langle f \rangle) \rangle\right) + \epsilon \nabla \cdot \left(\frac{1}{a} \langle v \partial_t f \rangle\right). \tag{1.2}$$

  Indeed, multiplying by $\dfrac{\epsilon}{a} v$ the equation $\partial_t f + \dfrac{1}{\epsilon} v \cdot \nabla f + \dfrac{a}{\epsilon^2} Sf = 0$ and taking the average over $\mathbb{S}^{d-1}$, using $\partial_t \langle f \rangle + \dfrac{1}{\epsilon} \langle v \cdot \nabla f \rangle = 0$, $\langle v \cdot \nabla Sf \rangle = \langle v \cdot \nabla f \rangle$ and $\langle v (v \cdot \nabla \langle f \rangle) \rangle = \frac{1}{d} \nabla \langle f \rangle$, one obtains, for any $t \geq 0$ and any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \partial_t \langle f \rangle \varphi \, dx + \frac{1}{d} \int_\Omega \frac{1}{a} \nabla \langle f \rangle \cdot \nabla \varphi \, dx + \int_\Omega \frac{1}{a} \langle v (v \cdot \nabla (f - \langle f \rangle) + \epsilon \, \partial_t f) \rangle \cdot \nabla \varphi \, dx = 0.$$

  Moreover, we prove that the boundary condition on $\langle f \rangle$ is small in some adequate norm with respect to ϵ. In the sequel, any estimate will be explicit with respect to ϵ.

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