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Numerical analysis

A successive constraint approach to solving parameter-dependent linear matrix inequalities

Une méthode de contraintes successives pour résoudre les inégalités matricielles linéaires paramétriques

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A R T I C L E   I N F O

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A B S T R A C T

We present a successive constraint approach that makes it possible to cheaply solve large-scale linear matrix inequalities for a large number of parameter values. The efficiency of our method is made possible by an offline/online decomposition of the workload. Expensive computations are performed beforehand, in the offline stage, so that the problem can be solved very cheaply in the online stage. We also extend the method to approximate solutions to semidefinite programming problems.

R É S U M É

Nous présentons une méthode de contraintes successives qui réduit le travail nécessaire pour résoudre les inégalités matricielles linéaires paramétriques de grande dimension. Une caractéristique importante de notre méthode est la décomposition hors ligne/en ligne du travail. Les calculs coûteux sont effectués à l'avance, hors ligne, pour nous permettre de résoudre le problème de manière très économique en ligne. La même méthode est aussi appliquée à l'approximation des solutions des problèmes d'optimisation SDP.

1. Introduction

Linear matrix inequalities (LMIs) are a general type of convex constraint that includes linear as well as quadratic constraints [11,12] and lead to very natural formulations of a large number of problems in control and systems theory [2,4]. They can be solved using a wide range of existing methods [1,5,6,10], but that can be expensive for large-scale problems. In particular, problems resulting from the discretization of partial differential equations can be extremely expensive due to their high dimensionality. The computations are even more expensive if parameter-dependent problems are considered and

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solutions are needed for a large number of parameter values. In that case traditional solution methods are extremely inefficient. We propose the construction of reduced-order models that take advantage of the parametric nature of the problem and allow us to very cheaply produce solutions for a large number of parameter values.

Let us introduce a finite-dimensional, bounded parameter domain $\mathcal{D} \in \mathbb{R}^p$ and the parameter-dependent LMI

$$F(x;\mu) := \sum_{q=1}^{Q_F} \left[\theta_q^0(\mu) + \theta_q^L(\mu)x\right] F_q \succeq 0. \tag{L}$$

Here $F(x;\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ is a symmetric matrix that depends on both the parameter $\mu \in \mathcal{D}$ and the decision variable $x \in \mathbb{R}^n$ and is composed of Q_F parameter-independent matrices $F_q \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$. The parameter dependencies of $F(x;\mu)$ are given by the functions $\theta^0(\cdot) : \mathcal{D} \to \mathbb{R}^{Q_F}$ and $\theta^L(\cdot) : \mathcal{D} \to \mathbb{R}^{Q_F \times n}$. We use subscripts to indicate components of a vector, such that $\theta_q^0(\mu)$ is the qth element of $\theta^0(\mu)$. Similarly, we write $\theta_q^L(\mu)$ to indicate the qth row of $\theta^L(\mu)$. The symbol $\succeq$ will be used in the sense that $P \succeq 0$ indicates that the symmetric matrix P is positive semi-definite.

The goal of this paper is to efficiently solve the following problems for a large number of parameter values $\mu \in \mathcal{D}$:

(i) the strict feasibility problem: find an $x \in \mathbb{R}^n$ such that $F(x;\mu) \succ 0$;
(ii) the semidefinite program (SDP):

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad c(\mu)^{\mathrm{T}}x \quad \text{subject to} \quad F(x;\mu) \succeq 0. \tag{S}$$

Rather than directly solving these problems for each new parameter value, we will solve them for only a small set of intelligently chosen parameter values. We will then use the resulting solutions to build a reduced-order model that can approximate the solution anywhere in $\mathcal{D}$.

The method that we propose can be viewed as a generalization of the successive constraint method (SCM) [3,7], which is often used in the field of reduced basis methods to evaluate stability constants [9]. This method will allow us to very cheaply determine feasible solutions for any $\mu \in \mathcal{D}$. That is made possible by decomposing the computational workload into offline and online stages. All expensive computations will be performed in advance, during the offline stage. During the online stage the cost to solve the problem for a new parameter value will be independent of the size of the original constraint, $\mathcal{N}$. In that way the computational cost of each new solution will remain cheap even if the original constraint has very large dimensions.

Our methods are applicable to a wide range of LMIs and can also be used to extend the applicability of SCM. In the context of reduced basis methods, applications could involve bounding stability constants with respect to parameter-dependent norms or the selection of Lyapunov functions for the computation of error bounds [8]. In Section 4 we present an example in which we optimize a system while ensuring that it remains stable.

2. Reduced-order modeling for strict feasibility

SCM was originally designed to approximate coercivity constants. We will apply a modified version of SCM to the coercivity constant

$$\alpha(x;\mu) := \inf_{v\in\mathbb{R}^{\mathcal{N}}} \frac{v^{\mathrm{T}}F(x;\mu)v}{v^{\mathrm{T}}F_S v}, \tag{1}$$

where $F_S \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ is a fixed symmetric positive-definite matrix. From the definition it is clear that $\alpha(x;\mu) \geq 0$ is equivalent to $F(x;\mu) \succeq 0$ for all symmetric positive definite matrices F_S. Nevertheless, an appropriate choice of F_S could be beneficial from a numerical point of view. If we are dealing with PDE discretizations, it can be advantageous to choose a matrix associated with an energy norm.

The first step in applying SCM is reformulating the coercivity constant as follows:

$$\alpha(x;\mu) = \inf_{y\in\mathcal{Y}} \left[\theta^0(\mu) + \theta^L(\mu)x\right]^{\mathrm{T}} y, \quad \text{where} \quad \mathcal{Y} := \left\{ y \in \mathbb{R}^{Q_F} \,\middle|\, y_q = \frac{v^{\mathrm{T}}F_q v}{v^{\mathrm{T}}F_S v},\ v \in \mathbb{R}^{\mathcal{N}} \right\}. \tag{2}$$

This formulation has the advantage that the complexity of the problem has been shifted to the definition of the set $\mathcal{Y}$. That allows us to compute lower and upper bounds for $\alpha(x;\mu)$ by approximating $\mathcal{Y}$.

A lower bound for $\alpha(x;\mu)$ can be derived by approximating $\mathcal{Y}$ from the outside. A bounded but primitive approximation for $\mathcal{Y}$ is given by

$$\mathcal{B}_Q := \prod_{i=1}^{Q_F} \left[\inf_{y\in\mathcal{Y}} y_q, \sup_{y\in\mathcal{Y}} y_q\right] \subset \mathbb{R}^{Q_F}. \tag{3}$$

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