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Partial differential equations/Optimal control

Observability estimates for the wave equation with rough coefficients ☆

Estimées d'observabilité pour l'équation des ondes avec des coefficients continus

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ABSTRACT

The goal of this note is to prove observability estimates for the wave equation with a density which is only continuous in the domain, and satisfies some multiplier-type condition only in the sense of distributions. Our main argument is that one can construct suitable approximations of such density by a sequence of smooth densities whose corresponding wave equations are uniformly observable. The end of the argument then consists in a rather standard passage to the limit.

RÉSUMÉ

Le but de cette note est de démontrer des estimées d'observabilité pour l'équation des ondes avec une densité continue dans le domaine, et qui satisfait une condition de type multiplicateur seulement au sens des distributions. Notre argument est essentiellement basé sur le fait que l'on peut alors construire des approximations convenables d'une telle fonction de densité par des fonctions régulières pour lesquelles les équations des ondes correspondantes sont uniformément observables. La preuve se termine alors par un passage à la limite relativement standard.

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1. Introduction

1.1. Setting and main result

The goal of this note is to study observability estimates for the wave equation with a density having low regularity. More precisely, we consider the wave equation

$$\begin{cases} \rho(x)\partial_t^2 u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u(0),\partial_t u(0)) = (u_0,u_1) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a smooth bounded domain of $\mathbb{R}^d$. The function $\rho = \rho(x)$ represents the density of the medium in which the wave of amplitude u propagates.

Our goal is to provide observability estimates for the wave equation (1.1) under weak regularity assumptions on the density ρ. Namely, we will assume the following regularity conditions:

- the density ρ is strictly positive and bounded in $\overline{\Omega}$: there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\forall x \in \Omega, \quad 0 < \rho_1 \le \rho(x) \le \rho_2, \tag{1.2}$$

- the density ρ is continuous in $\overline{\Omega}$:

$$\rho \in C^0(\overline{\Omega}). \tag{1.3}$$

The assumptions (1.2)–(1.3) are natural as they guarantee the well-posedness of the equation (1.1) for $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Indeed, in this framework, the work [19, Chapter 3, Sections 8–9] (see also [13] and the paper by F. Colombini et al. [7] for an overview on this question) shows that system (1.1) admits a unique solution $u(t,x)$ in the energy space $C^0([0,+\infty[, H_0^1(\Omega)) \cap C^1([0,+\infty[, L^2(\Omega))$. Moreover, the solution u to (1.1) has a constant energy as time evolves, i.e.

$$E[u](t) := \frac{1}{2}\int_\Omega \rho(x)|\partial_t u(t,x)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla_x u(t,x)|^2 \mathrm{d}x \tag{1.4}$$

is independent from the time t and satisfies

$$\forall t \in [0,T], \quad E[u](t) = E[u](0). \tag{1.5}$$

We will now consider an observability problem from an open subset ω which is an open neighborhood (in $\overline{\Omega}$) of an open subset Γ of the boundary satisfying the celebrated multiplier condition [18,17,14]. Namely, we assume that Γ satisfies the following condition:

$$\{x \in \partial\Omega, \text{ such that } x\cdot n_x > 0\} \subset \Gamma, \tag{1.6}$$

where for $x \in \partial\Omega$, n_x denotes the outward normal to the boundary $\partial\Omega$ at the point x and

$$\omega \text{ is an open neighborhood in } \overline{\Omega} \text{ of } \Gamma. \tag{1.7}$$

In order to simplify the presentation of our work, we will assume that the density ρ is defined on a smooth domain Ω_1 containing $\overline{\Omega}$ and satisfies assumptions (1.2)–(1.3) in Ω_1. Otherwise, one should assume that there exists a suitable extension of ρ to Ω_1 satisfying (1.2)–(1.3) and the appropriate conditions given afterwards.

We then assume that the density ρ satisfies the following condition:

$$\exists \alpha \in (0,2], \text{ such that } \quad x\cdot\nabla\rho(x) + (2-\alpha)\rho(x) \ge 0 \text{ in the sense of } \mathscr{D}'(\Omega_1). \tag{1.8}$$

We emphasize that condition (1.8) does not require any new regularity condition on ρ as the inequality in (1.8) only holds *in the sense of distributions*, meaning that:

$$\forall \varphi \in \mathscr{D}(\Omega_1), \text{ with } \varphi \ge 0, \quad \int_{\mathbb{R}^d} \rho(x)\left(-\mathrm{div}\,(x\varphi(x)) + (2-\alpha)\varphi(x)\right)\mathrm{d}x \ge 0. \tag{1.9}$$

Let us also note that this condition (1.8) describes some monotony condition on the density in the direction of the multiplier x, while it does not require any additional assumption in the tangential directions, see Section 1.3 for a more extensive discussion of this fact.

We are now in position to state our main result.