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Dynamical systems

Combinatorial models for spaces of cubic polynomials

Modèles combinatoires pour les espaces de polynômes cubiques

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A R T I C L E I N F O

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A B S T R A C T

W. Thurston constructed a combinatorial model of the Mandelbrot set $\mathcal{M}_2$ such that there is a continuous and monotone projection of $\mathcal{M}_2$ to this model. We propose the following related model for the space $\mathcal{MD}_3$ of critically marked cubic polynomials with connected Julia set and all cycles repelling. If $(P, c_1, c_2) \in \mathcal{MD}_3$, then every point z in the Julia set of the polynomial P defines a unique maximal finite set A_z of angles on the circle corresponding to the rays, whose impressions form a continuum containing z. Let $G(z)$ denote the convex hull of A_z. The convex sets $G(z)$ partition the closed unit disk. For $(P, c_1, c_2) \in \mathcal{MD}_3$ let c_1^* be the *co-critical point of* c_1. We tag the marked dendritic polynomial (P, c_1, c_2) with the set $G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Tags are pairwise disjoint; denote by $\mathcal{MD}_3^{\mathrm{comb}}$ their collection, equipped with the quotient topology. We show that tagging defines a continuous map from $\mathcal{MD}_3$ to $\mathcal{MD}_3^{\mathrm{comb}}$ so that $\mathcal{MD}_3^{\mathrm{comb}}$ serves as a model for $\mathcal{MD}_3$.

R É S U M É

W. Thurston a construit un modèle combinatoire de l'ensemble de Mandelbrot $\mathcal{M}_2$ tel qu'il y ait une projection monotone et continue de $\mathcal{M}_2$ sur ce modèle. En relation avec ceci, nous proposons le modèle lié suivant pour l'espace $\mathcal{MD}_3$ des polynômes cubiques à points critiques marqués, avec ensemble de Julia connexe et tous les cycles répulsifs. Si $(P, c_1, c_2) \in \mathcal{MD}_3$, alors chaque point z dans l'ensemble de Julia du polynôme P définit un unique ensemble fini maximal A_z d'angles sur le cercle correspondant aux rayons, dont les impressions forment un continuum contenant z. Soit $G(z)$ l'enveloppe convexe de A_z. Les ensembles convexes $G(z)$ définissent une partition du disque unité fermé. Pour $(P, c_1, c_2) \in \mathcal{MD}_3$, soit c_1^* le *point co-critique de* c_1. Nous balisons le polynôme dendritique marqué (P, c_1, c_2) avec l'ensemble $G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Les balises sont deux à deux disjointes ; désignons par $\mathcal{MD}_3^{\mathrm{comb}}$ leur collection, équipée de la topologie quotient. Nous

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montrons que le balisage définit une application continue de $\mathcal{MD}_3$ dans $\mathcal{MD}_3^{\text{comb}}$ de sorte que $\mathcal{MD}_3^{\text{comb}}$ est un modèle pour $\mathcal{MD}_3$.

1. Introduction

Let $\mathbb{D}$ be the open disk $\{z \in \mathbb{C} \mid |z| < 1\}$ in the plane, $\overline{\mathbb{D}}$ be its closure, and $\mathbb{S}$ be its boundary circle. Let P be a polynomial of degree d with connected Julia set $J(P)$. We write Φ_P for the conformal isomorphism between $\mathbb{C} \setminus \overline{\mathbb{D}}$ and the complement U of the filled Julia set $K(P)$ asymptotic to the identity at infinity. By a theorem of Carathéodory, if $J(P)$ is locally connected, then Φ_P can be extended to a continuous map $\overline{\Phi}_P : \mathbb{C} \setminus \mathbb{D} \to \overline{U}$, under which $\mathbb{S}$ maps onto $J(P)$. Define the *lamination generated by* P as the equivalence relation $\sim_P$ on $\mathbb{S}$ identifying points of $\mathbb{S}$ if and only if $\overline{\Phi}_P$ sends them to the same point of $J(P)$.

By Thurston [7], the map P restricted to its locally connected Julia set $J(P)$ is topologically conjugate to a self-mapping $f_{\sim_P}$ of the quotient space $\mathbb{S}/\sim_P = J_{\sim_P}$ induced by $z^d|_{\mathbb{S}} = \sigma_d$; denote this conjugacy by $\Psi_P : J(P) \to J_{\sim_P}$. The mapping $f_{\sim_P}$ is called a *topological polynomial*. The quotient map of $\mathbb{S}$ onto $\mathbb{S}/\sim_P$ is denoted by $\pi_{\sim_P}$. Given a point $z \in J(P)$, we let $G_P(z) = G(z)$ denote the convex hull of the set $\pi_{\sim_P}^{-1}(\Psi_P(z))$. In other words, we represent z by the point $\Psi_P(z)$ of the model topological Julia set $J_{\sim_P}$ and then take all angles associated with $\Psi_P(z)$ in the sense of the lamination $\sim_P$. By [7], for two points z and w, the sets $G(z)$ and $G(w)$ either coincide or are disjoint.

The *geolamination* (from *geodesic* or *geometric lamination*) of P is the collection of chords, each of which is an edge of the convex hull of a $\sim_P$-class. Geolaminations geometrically interpret and "topologize" laminations, reflecting limit transitions among them. Both laminations and their geolaminations can be defined intrinsically (without polynomials). Then some geolaminations will not directly correspond to an equivalence relation on $\mathbb{S}$, but the family of all geolaminations will be closed. This allows one to work with limits of geolaminations and limits of polynomials (which might have non-locally connected Julia sets).

Thurston [7] models polynomials by their geolaminations, and families of quadratic polynomials by families of quadratic geolaminations. He "tags" quadratic geolaminations with their *minors* which form the *quadratic minor geolamination* QML and generate the corresponding lamination $\sim_{\text{QML}}$. The quotient space $\mathbb{S}/\sim_{\text{QML}}$ models the boundary of the Mandelbrot set $\mathcal{M}_2$ (this is the set of all parameters c such that polynomials $z^2 + c$ have connected Julia set; it is also called the *quadratic connected locus*). The induced quotient space of $\overline{\mathbb{D}}$ serves as a model for $\mathcal{M}_2$. Conjecturally, it is homeomorphic to $\mathcal{M}_2$.

Call a polynomial with connected Julia set *dendritic* if all its periodic points are repelling. By [5], for any dendritic polynomial P, even if $J(P)$ is not locally connected, there is a lamination $\sim_P$ such that there exists a *monotone semi-conjugacy* Ψ_P between $P|_{J(P)}$ and the topological polynomial $f_{\sim_P}$. Thus the sets $G_P(z) = \pi_{\sim_P}^{-1}(\Psi_P(z))$ are well defined for every dendritic polynomial P and every point $z \in J(P)$. As we will see, these nice properties of *individual* dendritic polynomials result in nice properties of *families* of cubic dendritic polynomials.

Let $\mathcal{D}_2 \subset \mathcal{M}_2$ be the set of all parameters $c \in \mathcal{M}_2$ such that the polynomial $P_c(z) = z^2 + c$ is dendritic. Set $H_c = G_{P_c}(c)$, and let $\mathcal{H}$ stand for the collection of all sets H_c, $c \in \mathcal{D}_2$. We denote the union $\bigcup_{c \in \mathcal{D}_2} H_c$ by $\mathcal{H}^+$ (in what follows, for any collection $\mathcal{A}$ of sets, we write $\mathcal{A}^+$ for the union of all sets in $\mathcal{A}$). By a part of a major result of [7], for two parameter values c, $c' \in \mathcal{D}_2$, the sets H_c and $H_{c'}$ are either disjoint or equal. Moreover, the mapping $c \mapsto H_c$ from $\mathcal{D}_2$ to $\mathcal{H}$ is upper semi-continuous (if a sequence of dendritic parameters c_n converges to a dendritic parameter c, then $\limsup_{n\to\infty} G_{c_n} \subset G_c$). The set $\mathcal{D}_2$ (or, equivalently, the set of all dendritic quadratic polynomials defined up to a Moebius change of coordinates) projects continuously onto the quotient space of $\mathcal{H}^+$ defined by the partition of $\mathcal{H}^+$ into sets H_c with $c \in \mathcal{D}_2$.

We propose a related model for the space $\mathcal{MD}_3$ of *marked dendritic* cubic polynomials (P, c_1, c_2) with connected Julia set (c_1, c_2 are the critical points of P). Define the *co-critical* point associated with a critical point τ of P as the only point τ^* such that $P(\tau^*) = P(\tau)$, $\tau^* \neq \tau$ unless P has a unique critical point, in which case $\tau = \tau^*$. Then, with every marked dendritic cubic polynomial (P, c_1, c_2), we associate the corresponding *mixed tag* $\text{Tag}(P, c_1, c_2) = G(c_1^*) \times G(P(c_2)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. This defines the mixed tag $\text{Tag}(P, c_1, c_2)$ for *all marked dendritic cubic polynomials*. Our choice of tags is based on the following two requirements. Firstly, the tag of $\text{Tag}(P, c_1, c_2)$ must determine $\sim_P$. Secondly, different tags must be disjoint. It is easy to see that the post-critical tag $G(P(c_1)) \times G(P(c_2))$ does not determine $G(c_1)$ and $G(c_2)$. Hence it does not determine $\sim_P$ either. Co-critical tags $G(c_1^*) \times G(c_2^*)$ do not satisfy our requirements either since these tags may intersect without being the same (this happens, e.g., for unicritical polynomials). For this reason, we use mixed tags.

Theorem 1.1. *Mixed tags of elements in* $\mathcal{MD}_3$ *are disjoint or coincide so that sets* $\text{Tag}(P, c_1, c_2)$ *form a partition of the set* $\text{Tag}(\mathcal{MD}_3)^+ \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ *and generate the corresponding quotient space of* $\text{Tag}(\mathcal{MD}_3)^+$ *denoted by* $\mathcal{MD}_3^{\text{comb}}$*. Then* $\mathcal{MD}_3^{\text{comb}}$ *is a separable metric space and the map* $\text{Tag} : \mathcal{MD}_3 \to \mathcal{MD}_3^{\text{comb}}$ *is continuous.*

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