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Dynamical systems

On the Hamilton–Poisson realizations of the integrable deformations of the Maxwell–Bloch equations

Sur les réalisations Hamilton–Poisson des déformations intégrables des équations de Maxwell–Bloch

Cristian Lăzureanu

Department of Mathematics, Politehnica University of Timişoara, Piaţa Victoriei nr. 2, Timişoara, 300006, Romania

A R T I C L E I N F O

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A B S T R A C T

In this note, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations by modifying their constants of motions. We obtain two Hamilton–Poisson realizations of the new system. Moreover, we prove that the obtained system has infinitely many Hamilton–Poisson realizations. Particularly, we present a Hamilton–Poisson approach of the system obtained considering two concrete deformation functions.

R É S U M É

Dans cette Note, nous construisons des déformations intégrables des équations de Maxwell–Bloch en modifiant leurs constantes de mouvement. Nous obtenons deux réalisations Hamilton–Poisson du nouveau système. De plus, nous prouvons que le système obtenu admet des réalisations Hamilton–Poisson infiniment nombreuses. Nous présentons une approche Hamilton–Poisson du système obtenu en considérant deux fonctions particulières de déformation.

Version française abrégée

La construction des déformations intégrables a été étudiée dans des articles récents [1,5].

Dans cette Note, nous construisons des déformations intégrables des équations de Maxwell–Bloch [4]. Nous montrons qu'une telle déformation intégrable est un système bi-hamiltonien. De plus, nous analysons une déformation intégrable particulière des équations de Maxwell–Bloch. Plus précisément, nous établissons la stabilité de Lyapunov des points d'équi-

E-mail address: cristian.lazureanu@upt.ro.

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libre et nous prouvons l'existence d'orbites périodiques autour de ces points. Nous présentons les liens entre la fonction énergie-Casimir et les éléments dynamiques susmentionnés.

1. Introduction

In recent papers, the construction of integrable deformations of a given integrable system was studied. In [1], considering Poisson–Lie groups as deformations of Lie–Poisson (co)algebras, integrable deformations of both uncoupled and coupled versions of certain integrable types of Röossler and Lorenz systems were given. In [5], using the fact that the constants of motion uniquely determine the dynamical equations, integrable deformations of the Euler top were constructed.

In this paper, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations. The Maxwell–Bloch equations have significant importance in optics. These equations represent a model used to describe the interaction between laser light and a material sample composed of two-level atoms [4].

The paper is organized as follows.

In the second section, we prove that the constants of motion uniquely determine the Maxwell–Bloch equations, up to a parameterization of time. Using this property, we construct integrable deformations of the Maxwell–Bloch equations. In the third section, we show that such integrable deformation is a bi-Hamiltonian system. Moreover, this system has infinitely many Hamilton–Poisson realizations. In the last section, we analyze a particular integrable deformation of the Maxwell–Bloch equations. More precisely, we establish the Lyapunov stability of the equilibrium points, and we prove the existence of the periodic orbits around these points. We also present the connections between the energy-Casimir mapping and the aforementioned dynamical elements.

2. Integrable deformations of the Maxwell–Bloch equations

In this section, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations. We use the method considered in [5].

We recall that the equations

$$\dot{x} = y\,, \quad \dot{y} = xz\,, \quad \dot{z} = -xy \tag{1}$$

are called the three-dimensional real valued Maxwell–Bloch equations [4]. Moreover, two constants of motion in involution of system (1) are given by

$$I_1(x,y,z) = \frac{1}{2}x^2 + z\,, \quad I_2(x,y,z) = \frac{1}{2}y^2 + \frac{1}{2}z^2. \tag{2}$$

Let us prove that equations (1) are uniquely determined by these constants of motion, up to a parameterization of time. Indeed, differentiating the above constants of motion (2), we get

$$\dot{x} = -\frac{1}{x}\dot{z}\,, \quad \dot{y} = -\frac{z}{y}\dot{z}.$$

Considering $\dot{z} = -xyf$, where $f = f(t)$ is an arbitrary continuous function, we obtain

$$\dot{x} = yf\,, \quad \dot{y} = xzf\,, \quad \dot{z} = -xyf\,.$$

Using the transformation $t = t(\tau)$, where τ is the new time variable, given by $\tau = \int_0^t f(s)\,\mathrm{d}s$, it follows $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau} = yf \cdot \frac{1}{f} = y(\tau)$. Analogously, we obtain $\frac{\mathrm{d}y}{\mathrm{d}\tau} = x(\tau)z(\tau)$, $\frac{\mathrm{d}z}{\mathrm{d}\tau} = -x(\tau)y(\tau)$, as required.

The above property of equations (1) allows constructing integrable deformations of the Maxwell–Bloch equations altering their constants of motion. More precisely, let us consider the new constants of motion C_1 and C_2 given by

$$C_1(x,y,z) = \frac{1}{2}x^2 + z + \alpha(x,y,z) \text{ and } C_2(x,y,z) = \frac{1}{2}y^2 + \frac{1}{2}z^2 + \beta(x,y,z), \tag{3}$$

where α and β are arbitrary differentiable functions.

By (3), we have

$$x\dot{x} + \dot{z} + \frac{\partial\alpha}{\partial x}\dot{x} + \frac{\partial\alpha}{\partial y}\dot{y} + \frac{\partial\alpha}{\partial z}\dot{z} = 0\,, \quad y\dot{y} + z\dot{z} + \frac{\partial\beta}{\partial x}\dot{x} + \frac{\partial\beta}{\partial y}\dot{y} + \frac{\partial\beta}{\partial z}\dot{z} = 0,$$

or equivalent,

$$\left(x + \frac{\partial\alpha}{\partial x}\right)\dot{x} + \frac{\partial\alpha}{\partial y}\dot{y} = -\left(1 + \frac{\partial\alpha}{\partial z}\right)\dot{z}\,, \quad \frac{\partial\beta}{\partial x}\dot{x} + \left(y + \frac{\partial\beta}{\partial y}\right)\dot{y} = -\left(z + \frac{\partial\beta}{\partial z}\right)\dot{z}.$$