C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Calculus of variations

Pathological solutions to the Euler–Lagrange equation and existence/regularity of minimizers in one-dimensional variational problems ☆

Solutions pathologiques à l'équation d'Euler–Lagrange et existence/régularité des minimiseurs des problèmes variationnels en dimension un

Richard Gratwick [a], Aidys Sedipkov [b,c], Mikhail Sychev [b,c], Aris Tersenov [b,c]

[a] *Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK*
[b] *Laboratory of Differential Equations and Related Problems of Analysis, Sobolev Institute of Mathematics, Koptuyg Avenue, 4, Novosibirsk 630090, Russia*
[c] *Novosibirsk State University, Russia*

ARTICLE INFO

Article history:
Received 27 October 2016
Accepted 31 January 2017
Available online xxxx

Presented by John Ball

ABSTRACT

In this paper, we prove that if $L(x,u,v) \in C^3(R^3 \to R)$, $L_{vv} > 0$ and $L \geq \alpha|v| + \beta$, $\alpha > 0$, then all problems (1), (2) admit solutions in the class $W^{1,1}[a,b]$, which are in fact C^3-regular provided there are no pathological solutions to the Euler equation (5). Here $u \in C^3[c,d[$ is called a pathological solution to equation (5) if the equation holds in $[c,d[$, $|\dot{u}(x)| \to \infty$ as $x \to d$, and $||u||_{C[c,d]} < \infty$. We also prove that the lack of pathological solutions to the Euler equation results in the lack of the Lavrentiev phenomenon, see Theorem 9; no growth assumptions from below are required in this result.

RÉSUMÉ

Dans cette Note, nous démontrons que si $L(x,u,v) \in C^3(R^3 \to R)$, $L_{vv} > 0$ et $L \geq \alpha|v| + \beta$, $\alpha > 0$, alors tous les problèmes (1)–(2) admettent des solutions dans la classe $W^{1,1}[a,b]$, qui sont en fait C^3-régulières pourvu que l'équation d'Euler (5) n'ait pas de solution pathologique. Ici, une solution $u \in C^3[c,d[$ de (5) est dite pathologique si l'équation est satisfaite dans $[c,d[$, $|\dot{u}(x)| \to \infty$ lorsque $x \to d$ et $\|u\|_{C[c,d]} < \infty$. Nous montrons également (voir Théorème 9), que l'absence de solution pathologique à l'équation d'Euler entraîne

☆ This research was partially supported by the European Research Council/ERC Grant Agreement No. 291497 and by the grants RFBR N 15-01-08275 and 0314-2015-0012 from the Presidium of RAS.

E-mail address: masychev@math.nsc.ru (M. Sychev).

http://dx.doi.org/10.1016/j.crma.2017.01.020

l'absence de phénomène de Lavrentiev ; aucune hypothèse de croissance minimale n'est requise pour ce résultat.

In this paper we consider classical one-dimensional variational problems

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))dx \to \min, \tag{1}$$

$$u(a) = A, u(b) = B. \tag{2}$$

We assume that $L(x, u, v) : R \times R \times R \to R$ is of class C^3 and $L_{vv}(x, u, v) > 0$ everywhere. These assumptions on the integrand L will be regarded as *basic* throughout this article.

Under these assumptions, given a compact set $G \subset R^2$, we have that

$$L(x, u, v) \geq -\alpha|v| + \beta,\ \alpha > 0, \tag{3}$$

for $(x, u) \in G$, $v \in R$. Therefore, given a function $u \in W^{1,1}[a, b]$, we have that the function $L(\cdot, u(\cdot), \dot{u}(\cdot))$ is measurable and its negative part is integrable. Therefore, the integral $J(u)$ is defined and is either a finite value or $+\infty$.

In the case when the solution $u : [a, b] \to R$ is Lipschitz and $L \in C^1$ only, the Euler–Lagrange equation holds:

$$L_v(x, u(x), \dot{u}(x)) = \int_a^x L_u(t, u(t), \dot{u}(t))dt + c, \tag{4}$$

see, e.g., [2]. In case additionally L satisfies the basic assumptions, we have $u \in C^3[a, b]$ and the equation (4) can be resolved with respect to the second derivative of the function u:

$$u'' = \frac{L_u - L_{xv} - L_{uv}\dot{u}}{L_{vv}}, \tag{5}$$

which is the Euler equation, see again [2].

The basic update approach to studying the existence and regularity of minimizers is Tonelli's theory.

Theorem 1 *(Tonelli, [14]). If, in addition to the basic assumptions, $L(x, u, v)$ has superlinear growth in v, i.e. $L \geq \theta(v)$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$, then each problem (1), (2) admits a solution in the class $W^{1,1}[a, b]$.*

Theorem 2 *(Tonelli, [15]; Ball–Mizel [1]). If in the problem (1), (2) with L satisfying the basic assumptions, there exists a solution u_0 in the class $W^{1,1}[a, b]$, then each such solution has everywhere a classical derivative (possibly infinite) which is continuous as a function with values in $\bar{R} = R \cup \{-\infty, \infty\}$. In particular, u_0 is of class C^3 in an open set of full measure where it also satisfies the Euler equation (5).*

Corollary 3. *Suppose L satisfies the conditions of Theorem 1. Suppose also that there are no pathological solutions to the Euler equation (5) on the interval $[a, b]$, i.e. ones such that $u \in C^3[c, d[$ ($[c, d[\subset [a, b]$ and possibly $d < c$), u satisfies (5) in $[c, d[$, and $|\dot{u}(x)| \to \infty$ as $x \to d$. Then each problem (1), (2) admits a solution in the class $W^{1,1}[a, b]$ and all such solutions are C^3-regular functions.*

Therefore the assumptions of Tonelli's theory are the basic assumptions on L, the superlinear growth of $L(x, u, v)$ in v, and the lack of pathological solutions to the Euler equation (5). Superlinear growth is needed to state weak compactness in $W^{1,1}$ of minimizing sequences; existence then follows from lower semicontinuity of the functional J with respect to weak convergence in $W^{1,1}$, which is guaranteed by convexity of L in v, see, e.g., [13] for a modern proof of this fact. This existence/regularity theory became the basic one in the literature, see, e.g., [3]. Singular solutions to minimization problems were constructed comparably recently, see the papers of Ball–Mizel [1], Clarke–Vinter [4], Davie [5], Sychev [10,11], Gratwick [6,7].

The discovery of this paper is that the lack of pathological solutions to the Euler equation (5) is by itself sufficient both for existence and regularity of minimizers in the class $W^{1,1}$. The following theorem holds.

Theorem 4. *Let L satisfy the basic assumptions and let*

$$L(x, u, v) \geq \alpha|v| + \beta,\ \alpha > 0.$$