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Number theory

Growth of class numbers of positive definite ternary unimodular Hermitian lattices over imaginary number fields

Croissance du nombre de classes de réseaux hermitiens unimodulaires sur un corps quadratique imaginaire

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ABSTRACT

We give a lower bound for class numbers of unimodular ternary Hermitian lattices over imaginary quadratic fields. This shows that class numbers of unimodular Hermitian lattices grow infinitely as the field discriminants grow.

RÉSUMÉ

Nous donnons une borne inférieure pour le nombre de classes de réseaux ternaires hermitiens unimodulaires sur corps quadratique imaginaire. Cela montre que le nombre de classes de réseaux unimodulaires hermitienne tend vers l'infini avec le discriminant du corps.

1. Introduction

As a variant of quadratic forms, we can define Hermitian lattices over rings of integers in imaginary quadratic fields. Also, thanks to many mathematicians, we can think about the local structure of Hermitian lattices [4,12,1]. Hasse–Minkowski's theorem for quadratic forms guarantees that local representations over $\mathbb{Z}_p$ for all prime spots p imply the global representation over $\mathbb{Q}$. But, in general, it does not imply the representation over $\mathbb{Z}$. Landherr's theorem is a similar result for Hermitian lattices [8]. But, it does not imply the representation over the ring $\mathcal{O}$ of integers, either. The measure of non-representability over $\mathcal{O}$ is presented by the number of non-isometric Hermitian lattices that are locally isometric to the given Hermitian

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lattice. This number is called the *class number* of that lattice. We give a lower bound for ternary unimodular Hermitian lattices involving discriminants of fields.

The exact formula for class numbers of binary or ternary unimodular Hermitian lattices was given by Hashimoto and Koseki [2, Main Theorems 5.1, 5.2]. Their formula was expressed by using the field class numbers, the numbers of prime divisors, Dirichlet characters, Bernoulli numbers, as well as field discriminants. So, it is hard to see the bounds for the class numbers by using Hashimoto–Koseki's formula.

Our formula for the lower bound involves only field discriminants, so that it is easy to calculate the formula, although the lower bound is not accurate. Besides, the inequality for the lower bound shows that the class number grows asymptotically according to the discriminant.

2. Preliminaries

Let $E=\mathbb{Q}(\sqrt{-m})$ for a square-free positive integer m and $\mathcal{O}=\mathcal{O}_E=\mathbb{Z}[\omega]$ be its ring of integers, where $\omega=\omega_m=\sqrt{-m}$ or $\frac{1+\sqrt{-m}}{2}$ if $m\equiv 1,2$ or $3 \pmod 4$, respectively.

The localization is defined according to the behaviors of primes. For a prime p we define $E_p:=E\otimes_{\mathbb{Q}}\mathbb{Q}_p$. Then the ring $\mathcal{O}_p$ of integers of E_p is defined as $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p$. If p is inert or ramifies in E, then $E_p=\mathbb{Q}_p(\sqrt{-m})$ and $\alpha\otimes\beta=\alpha\beta$ with $\alpha\in E$ and $\beta\in\mathbb{Q}_p$. If p splits in E, then $E_p=\mathbb{Q}_p\times\mathbb{Q}_p$ and $\alpha\otimes\beta=(\alpha\beta,\overline{\alpha}\beta)$, where $\bar{\cdot}$ is the canonical involution. Thus E_p allows the unique involution $\overline{\alpha\otimes\beta}=\overline{\alpha}\otimes\beta$ [12,1].

Definition 2.1. Let $F=E$ or E_p. A *Hermitian space* is a finite-dimensional vector space V over F equipped with a sesquilinear map $H:V\times V\to F$ satisfying the following conditions:

1. $H(\mathbf{x},\mathbf{y})=\overline{H(\mathbf{y},\mathbf{x})}$,
2. $H(a\mathbf{x},\mathbf{y})=aH(\mathbf{x},\mathbf{y})$,
3. $H(\mathbf{x}_1+\mathbf{x}_2,\mathbf{y})=H(\mathbf{x}_1,\mathbf{y})+H(\mathbf{x}_2,\mathbf{y})$.

We simply denote $H(\mathbf{v},\mathbf{v})$ by $H(\mathbf{v})$ and call it the *(Hermitian) norm* of $\mathbf{v}$.

Definition 2.2. Let $R=\mathcal{O}$ or $\mathcal{O}_p$. A *Hermitian R-lattice*, or briefly a lattice, is an R-module L in a Hermitian space (V,H) with $H(L,L)\subseteq R$. If $H(\mathbf{v})>0$ for any nonzero vector $\mathbf{v}\in L$, then we call L *positive definite*.

If L is free with a basis $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$, then we define

$$M_L:=\left[H(\mathbf{v}_i,\mathbf{v}_j)\right]_{n\times n},$$

and call it the *Gram matrix* of L. We often identify M_L with the lattice L. If M_L is diagonal, we simply write $L=\langle a_1,\dots,a_n\rangle$, where $a_i=H(\mathbf{v}_i)$ for $i=1,2,\dots,n$. The determinant of M_L is called the *discriminant* of L, denoted by dL. If dL is a unit, we call L *unimodular*. We define the *rank* of L by $\operatorname{rank} L:=\dim_F F\otimes L$. For unexplained terminology and for more information, see [9].

3. Lower bounds for class numbers of unimodular Hermitian lattices

If a positive definite Hermitian lattice represents every positive definite binary Hermitian lattice, we call it 2-universal. In [6], we classified all ternary and quaternary Hermitian lattices that are 2-universal.

$$\begin{aligned}
&\mathbb{Q}(\sqrt{-1}): && \langle 1,1,1\rangle, \quad \langle 1,1\rangle\perp\begin{bmatrix}2&1\\1&2\end{bmatrix}\\
&\mathbb{Q}(\sqrt{-2}): && \langle 1,1\rangle\perp\begin{bmatrix}2&-1+\omega_2\\-1+\overline{\omega}_2&2\end{bmatrix}\\
&\mathbb{Q}(\sqrt{-3}): && \langle 1,1,1\rangle, \quad \langle 1,1,2\rangle\\
&\mathbb{Q}(\sqrt{-7}): && \langle 1,1,1\rangle\\
&\mathbb{Q}(\sqrt{-11}): && \langle 1,1\rangle\perp\begin{bmatrix}2&\omega_{11}\\\overline{\omega}_{11}&2\end{bmatrix}.
\end{aligned}$$

We obtain a lower bound for class numbers of unimodular Hermitian lattices by investigating the ranks of 2-universal lattices. Denote the minimal rank of 2-universal Hermitian lattices over $\mathbb{Q}(\sqrt{-m})$ by $u_2(-m)$. We know that $u_2(-1)=3$, $u_2(-2)=4$, $u_2(-3)=3$, $u_2(-7)=3$, and $u_2(-11)=4$ from the above list.

Lemma 3.1. *Let L be a ternary unimodular lattice over an imaginary quadratic field E. Then L is locally 2-universal.*

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