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Lie algebras

A remark on boundary level admissible representations

Une remarque sur les représentations admissibles de niveau limite

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ABSTRACT

We point out that it is immediate by our character formula that in the case of a *boundary level* the characters of admissible representations of affine Kac–Moody algebras and the corresponding W-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

RÉSUMÉ

Nous remarquons la conséquence suivante de notre formule de caractères. Pour un niveau limite, les caractères d'une représentation admissible d'une algèbre de Kac–Moody affine ainsi que de la W-algèbre correspondante s'expriment comme des produits de formes de Jacobi $\vartheta_{11}(\tau, z)$.

Recently a remarkable map between 4-dimensional superconformal field theories and vertex algebras has been constructed [1]. This has led to new insights in the theory of characters of vertex algebras. In particular it was observed that in some cases these characters decompose in nice products [10,8].

The purpose of this note is to explain the latter phenomena. Namely, we point out that it is immediate by our character formula [5,6] that in the case of a *boundary level* the characters of admissible representations of affine Kac–Moody algebras and the corresponding W-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

We would like to thank Wenbin Yan for drawing our attention to this question.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Let $Q = \mathbb{Z}\Delta$ be the root lattice and let $Q^* = \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{Z}$ for all $\alpha \in \Delta\}$ be the dual lattice. Let $\Delta_+ \subset \Delta$ be a subset of positive roots, let $\{\alpha_1, \dots, \alpha_\ell\}$ be the set of simple roots and let ρ be half of the sum of positive roots. Let W be the Weyl group. Let $(\,.\,|\,.\,)$ be the invariant symmetric bilinear form on $\mathfrak{g}$, normalized by the condition $(\alpha|\alpha) = 2$ for a long root α, and let $h^\vee$ be the dual Coxeter number ($= \frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$). We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(\,.\,|\,.\,)$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the associated with $\mathfrak{g}$ affine Kac–Moody algebra (see [3] for details), let $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$ be its Cartan subalgebra. We extend the symmetric bilinear form $(\,.\,|\,.\,)$ from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}d) = 0$, $(K|K) = 0$,

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$(d|d) = 0$, $(d|K) = 1$, and we identify $\widehat{\mathfrak{h}}^*$ with $\widehat{\mathfrak{h}}$ using this form. Then d is identified with the 0th fundamental weight $\Lambda_0 \in \widehat{\mathfrak{h}}^*$, such that $\Lambda_0|_{\mathfrak{g}[t,t^{-1}]+\mathbb{C}d} = 0$, $\Lambda_0(K) = 1$, and K is identified with the imaginary root $\delta \in \widehat{\mathfrak{h}}^*$. Then the set of real roots of $\widehat{\mathfrak{g}}$ is $\hat{\Delta}^{\mathrm{re}} = \{\alpha + n\delta | \alpha \in \Delta, n \in \mathbb{Z}\}$ and the subset of positive real roots is $\hat{\Delta}^{\mathrm{re}}_+ = \Delta_+ \cup \{\alpha + n\delta | \alpha \in \Delta, n \in \mathbb{Z}_{\geq 1}\}$. Let $\hat{\rho} = h^\vee \Lambda_0 + \rho$. Let

$$\hat{\Pi}_u = \{u\delta - \theta, \alpha_1, \dots, \alpha_\ell\},$$

where $\theta \in \Delta_+$ is the highest root, so that $\hat{\Pi}_1$ is the set of simple roots of $\widehat{\mathfrak{g}}$. For $\alpha \in \hat{\Delta}^{\mathrm{re}}$ one lets $\alpha^\vee = 2\alpha/(\alpha|\alpha)$. Finally, for $\beta \in Q^*$ define the translation $t_\beta \in \operatorname{End} \widehat{\mathfrak{h}}^*$ by

$$t_\beta(\lambda) = \lambda + \lambda(K)\beta - ((\lambda|\beta) + \frac{1}{2}\lambda(K)|\beta|^2)\delta.$$

Given $\Lambda \in \widehat{\mathfrak{h}}^*$ let $\hat{\Delta}^\Lambda = \{\alpha \in \hat{\Delta}^{\mathrm{re}} \,|\, (\Lambda|\alpha^\vee) \in \mathbb{Z}\}$. Then Λ is called an *admissible* weight if the following two properties hold:

(i) $(\Lambda + \widehat{\rho}|\alpha^\vee) \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \hat{\Delta}_+$,
(ii) $\mathbb{Q}\hat{\Delta}^\Lambda = \mathbb{Q}\hat{\Delta}^{\mathrm{re}}$.

If instead of (ii) a stronger condition holds:

(ii)′ $\varphi(\hat{\Delta}^\Lambda) = \hat{\Delta}^{\mathrm{re}}$ for a linear isomorphism $\varphi : \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*$,

then Λ is called a *principal* admissible weight. In [6] the classification and character formulas for admissible weights are reduced to that for principal admissible weights. The latter are described by the following proposition.

Proposition 1. *[6] Let Λ be a principal admissible weight and let $k = \Lambda(K)$ be its level. Then*

(a) k is a rational number with denominator $u \in \mathbb{Z}_{\geq 1}$, such that

$$k + h^\vee \geq \frac{h^\vee}{u} \text{ and } \gcd(u, h^\vee) = \gcd(u, r^\vee) = 1, \tag{1}$$

where $r^\vee = 1$ for $\mathfrak{g}$ of type A-D-E, $= 2$ for $\mathfrak{g}$ of type B, C, F, and $= 3$ for $\mathfrak{g} = G_2$.

(b) All principal admissible weights are of the form

$$\Lambda = (t_\beta y).(\Lambda^0 - (u-1)(k + h^\vee)\Lambda_0), \tag{2}$$

where $\beta \in Q^*$, $y \in W$ are such that $(t_\beta y)\hat{\Pi}_u \subset \hat{\Delta}_+$, Λ^0 is an integrable weight of level $u(k + h^\vee) - h^\vee$, and dot denotes the shifted action: $w.\Lambda = w(\Lambda + \widehat{\rho}) - \widehat{\rho}$.

(c) For $\mathfrak{g} = s\ell_N$ all admissible weights are principal admissible.

Recall that the normalized character of an irreducible highest weight $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level $k \neq -h^\vee$ is defined by

$$\mathrm{ch}_\Lambda(\tau, z, t) = q^{m_\Lambda} \mathrm{tr}_{L(\Lambda)} \mathrm{e}^{2\pi i h}$$

where

$$h = -\tau d + z + tK,\ z \in \mathfrak{h},\ \tau, t \in \mathbb{C},\ \mathrm{Im}\,\tau > 0,\ q = \mathrm{e}^{2\pi i \tau}, \tag{3}$$

and $m_\Lambda = \frac{|\Lambda + \widehat{\rho}|^2}{2(k+h^\vee)} - \frac{\dim \mathfrak{g}}{24}$ (the normalization factor q^{m_Λ} "improves" the modular invariance of the character).

In [6], the characters of the $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ for arbitrary admissible Λ were computed, see Theorem 3.1, or formula (3.3) there for another version in case of a principal admissible Λ. In order to write down the latter formula, recall the normalized affine denominator for $\widehat{\mathfrak{g}}$:

$$\hat{R}(h) = q^{\frac{\dim \mathfrak{g}}{24}} e^{\widehat{\rho}(h)} \prod_{n=1}^{\infty} (1 - q^n)^\ell \prod_{\alpha \in \Delta_+} (1 - \mathrm{e}^{\alpha(z)} q^n)(1 - \mathrm{e}^{-\alpha(z)} q^{n-1}).$$

In coordinates (3) this becomes:

$$\hat{R}(\tau, z, t) = (-i)^{|\Delta_+|} \mathrm{e}^{2\pi i h^\vee t} \eta(\tau)^{\frac{1}{2}(3\ell - \dim \mathfrak{g})} \prod_{\alpha \in \Delta_+} \vartheta_{11}(\tau, \alpha(z)), \tag{4}$$

where