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Mathematical analysis/Partial differential equations

Dispersive estimates for the wave equation inside cylindrical convex domains: A model case ☆

Estimation de dispersion pour les ondes dans un convexe : le cas modèle

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ABSTRACT

In this work, we will establish local in time dispersive estimates for solutions to the model-case Dirichlet wave equation inside a cylindrical convex domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega \neq \emptyset$. Let us recall that dispersive estimates are key ingredients to prove Strichartz estimates. Nonoptimal Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair–Smith–Sogge [1,2]. Better estimates in strictly convex domains have been obtained in [4]. Our case of cylindrical domains is an extension of the result of [4] in the case where the curvature radius ≥ 0 depends on the incident angle and vanishes in some directions.

RÉSUMÉ

Dans ce travail, nous allons établir des estimations de dispersion locales en temps pour les solutions de l'équation des ondes dans un domaine cylindrique convexe $\Omega \subset \mathbb{R}^3$ à bord C^∞ $\partial\Omega \neq \emptyset$. Les estimations de dispersion sont classiquement utilisées pour prouver les estimations de Strichartz. Dans un domaine Ω général, des estimations de Strichartz non optimales ont été démontrées par Blair–Smith–Sogge [1,2]. De meilleures estimations ont été prouvées dans [4] lorsque Ω est strictement convexe. Le cas des domaines cylindriques que nous considérons ici généralise les resultats de [4] dans le cas où la courbure ≥ 0 dépend de l'angle d'incidence et s'annule dans certaines directions.

Version française abrégée

Soit $\Omega = \{x \geq 0, (y, z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Nous établissons des estimations de dispersion en temps petit pour l'équation des ondes dans Ω avec condition de Dirichlet.

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$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0$$

avec $P(t, x, y, z, \partial_t, \partial_x, \partial_y, \partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2)$, $u = u(t, x, y, z)$, et pour $a \in \Omega$, $\delta_a = \delta_{x=a, y=0, z=0}$. Le problème est local près de chaque point du bord. Les phénomènes nouveaux apparaissent pour $0 < a \leq 1$ petit. Nous utilisons les notations $\tau = \frac{h}{i}\partial_t$, $\eta = \frac{h}{i}\partial_y$, $\xi = \frac{h}{i}\partial_x$, $\zeta = \frac{h}{i}\partial_z$.

Résultats

On note $G_a(t, x, y, z)$ la fonction de Green associée à (1). La fonction χ appartient à $C_0^\infty(]0, \infty[)$ et est égale à 1 sur l'intervalle $[1, 2]$. Nous prouvons le théorème suivant.

Théorème 0.1. *Il existe $C > 0$ tel que, pour tout $a \in]0, 1]$, $h \in]0, 1]$, $t \in [-1, 1] \setminus \{0\}$, on a*

$$\|\chi(hD_t)G_a(t, x, y, z)\|_{L^\infty} \leq Ch^{-3} \min\left(1, \left(\frac{h}{|t|}\right)^{1/2} \gamma\big(t, h, \sup(x, a)\big)\right),$$

avec

$$\gamma(t, h, b) = \begin{cases} \left(\frac{h}{|t|}\right)^{1/2} + b^{1/8}h^{1/4} & \text{si } b \geq h^{2/3-\epsilon}, \text{ pour tout } \epsilon > 0 \text{ petit}, \\ h^{1/4} + \left(\frac{h}{|t|}\right)^{1/3} & \text{si } b \leq h^{1/2}. \end{cases}$$

1. Introduction

We will study the following model case. Let $\Omega = \{x \geq 0, (y, z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Our goal is to establish local in time dispersive estimates for solutions to the linear Dirichlet wave equation inside Ω with smooth boundary $\partial\Omega$:

$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0 \tag{1}$$

with $P(t, x, y, z, \partial_t, \partial_x, \partial_y, \partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2)$, $u = u(t, x, y, z)$, and for $a \in \Omega$, $\delta_a = \delta_{x=a, y=0, z=0}$ (the Dirac distribution). The problem is local near any points on the boundary. We are interested only in highly reflected waves and near points where the order of tangency is infinity whose source points are located at a small distance $0 < a \leq 1$ to the boundary. This gives us interesting phenomena such as caustics near the boundary for such domains. We will use the notation $\tau = \frac{h}{i}\partial_t$, $\eta = \frac{h}{i}\partial_y$, $\xi = \frac{h}{i}\partial_x$, $\zeta = \frac{h}{i}\partial_z$. We recall that the dispersive estimates for the free wave in $\mathbb{R}^3$ read as follows:

$$\|\chi(hD_t)e^{\pm it\sqrt{\Delta}}(\delta_a)\|_{L^\infty} \leq Ch^{-3} \min\left(1, \frac{h}{|t|}\right).$$

1.1. Main results

In the following, we prove dispersive estimates in cylindrical domains. The main results are obtained for appropriately localized Green function in each region corresponding to different values of η and we prove that the estimates are worse than that of the free case. To be more precise, the following theorem holds true.

Theorem 1.1. *There exists C such that for every $a \in]0, 1]$, every $h \in]0, 1]$, every $t \in [-1, 1] \setminus 0$, the following holds:*

$$\|\chi(hD_t)G_a(t, x, y, z)\|_{L^\infty} \leq Ch^{-3} \min\left(1, \left(\frac{h}{|t|}\right)^{1/2} \gamma\big(t, h, \sup(x, a)\big)\right), \tag{2}$$

with

$$\gamma(t, h, b) = \begin{cases} \left(\frac{h}{|t|}\right)^{1/2} + b^{1/8}h^{1/4} & \text{if } b \geq h^{2/3-\epsilon}, \text{ for any } \epsilon > 0 \text{ small}, \\ h^{1/4} + \left(\frac{h}{|t|}\right)^{1/3} & \text{if } b \leq h^{1/2}. \end{cases}$$

2. Idea of the proof

The main idea to prove the theorem is to construct a suitable local parametrix together with the Airy–Poisson summation formula. In this way, the parametrix is represented as a sum over eigenmodes, which is used to prove the estimates for $a \leq h^{1/2}$. On the other hand, the parametrix is represented as a sum over multiple reflections, which is used to prove the estimates for $a \geq h^{2/3-\epsilon}$.