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Differential geometry

Variation of Laplace spectra of compact “nearly” hyperbolic surfaces

Variation du spectre de Laplace des surfaces compactes « presque » hyperboliques

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ABSTRACT

We use the real analyticity of the Ricci flow with respect to time proved by B. Kotschwar to extend a result of P. Buser, namely, we prove that the Laplace spectra of negatively curved compact orientable surfaces having the same genus $\gamma \geq 2$, the same area and the same curvature bounds vary in a “controlled way”, of which we give a quantitative estimate in our main theorem. The basic technical tool is a variational formula that provides the derivative of an eigenvalue branch under the normalized Ricci flow. In a related manner, we also observe how the above-mentioned real analyticity result can lead to unexpected conclusions concerning the spectral properties of generic metrics on a compact surface of genus $\gamma \geq 2$.

RÉSUMÉ

Nous utilisons l’analyticité réelle du flot de Ricci par rapport au temps, démontrée par B. Kotschwar, pour étendre un résultat de P. Buser. Précisément, nous montrons que le spectre de Laplace des surfaces compactes, orientables, de courbure négative, de même genre $\gamma \geq 2$, même aire et mêmes bornes pour la courbure, varie de « façon contrôlée ». Nous donnons une estimation quantitative de cette variation dans notre théorème principal. Notre outil technique de base est une formule variationnelle donnant la dérivée d’une branche de valeur propre sous l’action du flot de Ricci normalisé. Par analogie, nous indiquons comment le résultat d’analyticité réelle ci-dessus peut conduire à des conclusions inattendues sur les propriétés du spectre des métriques génériques sur une surface compacte, de genre $\gamma \geq 2$.

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1. Normalized Ricci flow and evolution of the spectrum

1.1. Introduction

Consider two compact orientable negatively curved surfaces M_1 and M_2 of same genus $\gamma \geq 2$ and same area A, such that their scalar curvatures R_1 and R_2 (respectively) satisfy $\alpha < R_i < \beta < 0$ for $i = 1, 2$. Now, consider the compact surfaces N_i obtained by scaling the metrics on M_i, so that the N_i have average scalar curvature -2. Let N_i flow to hyperbolic surfaces S_i in the limit under the normalized Ricci flow. In this note, our chief aim is to prove the following.

Theorem 1.1. *Let δ be the distance between S_1 and S_2 in the Teichmüller space $\mathcal{T}_g$. Then, we have*

$$e^{-\frac{\alpha}{r}-\frac{r}{\beta}-4\delta}\lambda_n(M_2) \leq \lambda_n(M_1) \leq e^{\frac{\alpha}{r}+\frac{r}{\beta}+4\delta}\lambda_n(M_2), n \in \mathbb{N},$$

where r denotes the average scalar curvatures of M_i, $i = 1, 2$ and $\lambda_n(M)$ represent the Laplace eigenvalues of M.

For a definition of the distance in the Teichmüller space alluded to above, see Definition 2.3 in Subsection 2.1.

1.2. Background: Ricci flow facts

The Ricci flow program was introduced by Hamilton in [8]; the main idea is to take an initial metric and flow it into "nicer" metrics, at least with respect to curvature properties, according to the equation

$$\partial_t g = -2\,\mathrm{Ric}\, g, \tag{1}$$

where g denotes the (time-dependent) metric on M, and Ric the Ricci curvature tensor associated with the said metric. Hamilton and later DeTurck [6] provided proofs of the short-time existence of (1). It has also been proved that the Ricci flow (henceforth abbreviated RF) on a compact manifold can continue as long as the Riemannian curvature tensor does not explode. To balance this blow-up phenomenon, one sometimes uses the so-called "normalized Ricci flow" (henceforth abbreviated NRF), which is defined by

$$\partial_t g = -2\,\mathrm{Ric}\, g + \frac{2}{n} r g, \tag{2}$$

where R is the scalar curvature and $r = \frac{1}{\mathrm{Vol}\, M}\int_M R\,\mathrm{d}V$ is the average scalar curvature of the manifold M of dimension n. One main difference between (1) and (2) is that (2) rescales the volume of the manifold at every step, so that the volume remains constant throughout the flow. Using the NRF on surfaces, Hamilton proved the following theorem.

Theorem 1.2. *If (M, g_0) is a closed Riemannian surface, then there exists a unique solution g_t of the NRF*

$$\partial_t g = (r - R)g, \ \ g(0) = g_0. \tag{3}$$

The solution exists for all time. As $t \to \infty$, the metrics g_t converge uniformly in any C^k-norm to a smooth metric g_∞ of constant curvature.

If T is the maximal time for the existence of (1), by a well-known result of Bando (see [1]), a solution $(M, g(t))$, $t \in (0, T]$ is real analytic in space when M is compact. This was improved upon by Kotschwar [11], who provided sufficient conditions for a solution $(M, g(t))$ to be real analytic in both space and time when (M, g_0) is complete. He proved the following theorem.

Theorem 1.3. *Suppose (M, g_0) is complete and $g(t)$ is a smooth solution to (1) satisfying*

$$\sup_{M\times[0,\Omega]} |\mathrm{Rm}(x,t)| \leq C. \tag{4}$$

Then the map $g : (0, \Omega) \to X$ is real-analytic where X denotes the Banach space $BC(T_2(M))$ equipped with the supremum norm $\|.\|_{g(0)}$ relative to $g(0)$.

It is a natural question to ask what happens to the spectrum of the Laplacian under the Ricci flow. Theorem 1.3, in conjunction with Kato's analytic perturbation theory, tells us that the eigenvalues and eigenfunctions of the Laplacian vary real analytically in time as long as (4) is satisfied. Let us also remark here that without Kotschwar's result, we are assured of the twice differentiability of the eigenvalues from very general perturbative arguments. For more details on the variation of eigenvalues and eigenvectors of a one-parameter family of unbounded self-adjoint operators with common domain of definition and compact resolvent on a variety of regularity scales, see [12].

We note that Kotschwar's result is valid a priori for Ricci flows. But using the correspondence between RF and NRF, we can establish that the NRF is real analytic in time if the RF is. If the NRF is written as