C. R. Acad. Sci. Paris, Ser. I 355 (2017) 226–231

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical problems in mechanics/Differential geometry

New nonlinear estimates for surfaces in terms of their fundamental forms

Nouvelles estimations pour des surfaces en fonction de leurs formes fondamentales

Philippe G. Ciarlet[a], Maria Malin[a], Cristinel Mardare[b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Sorbonne Universités, Université Pierre-et-Marie-Curie, Laboratoire Jacques-Louis-Lions, 4, place Jussieu, 75005 Paris, France*

A R T I C L E I N F O

Article history:
Received 6 January 2017
Accepted 6 January 2017
Available online 1 February 2017

Presented by Philippe G. Ciarlet

A B S T R A C T

We establish several estimates of the distance between two surfaces immersed in the three-dimensional Euclidean space in terms of the distance between their fundamental forms, measured in various Sobolev norms. These estimates, which can be seen as nonlinear versions of linear Korn inequalities on a surface appearing in the theory of linearly elastic shells, generalize in particular the nonlinear Korn inequality established in 2005 by P. G. Ciarlet, L. Gratie, and C. Mardare.

R É S U M É

Nous établissons plusieurs majorations de la distance entre deux surfaces immergées dans l'espace euclidien tridimensionnel en fonction de la distance entre leurs formes fondamentales, mesurée à l'aide de diverses normes de Sobolev. Ces estimations, qui peuvent être vues comme des versions non linéaires des inégalités de Korn linéaires sur une surface apparaissant dans la théorie de coques linéairement élastiques, généralisent en particulier l'inégalité de Korn non linéaire sur une surface établie en 2005 par P. G. Ciarlet, L. Gratie et C. Mardare.

1. Introduction

The various notions and notations, notably from the differential geometry of surfaces, used in this introduction are defined in Sect. 2.

E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), mmalin2-c@my.cityu.edu.hk (M. Malin), mardare@ann.jussieu.fr (C. Mardare).

http://dx.doi.org/10.1016/j.crma.2017.01.002

A *nonlinear Korn inequality on a surface* is an inequality that estimates the deformation of a surface in terms of the variation of the fundamental forms of the surface induced by this deformation.

We establish here such inequalities for *deformations with as little regularity as possible*, that is, just enough to define the fundamental forms of the deformed surface in appropriate Lebesgue spaces L^q. This is motivated by the theory of nonlinearly elastic shells, in particular by the well-known nonlinear Koiter shell model, where the deformations with finite energy are those whose first two fundamental forms have covariant components in the Lebesgue space L^2.

More specifically, let $\omega \subset \mathbb{R}^2$ be a domain, let γ_0 be a non-empty relatively open subset of the boundary of ω, and let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be a sufficiently smooth immersion. Consider a shell with middle surface $\boldsymbol{\theta}(\overline{\omega}) \subset \mathbb{R}^3$ and half-thickness $h > 0$, made of a homogeneous and isotropic nonlinearly elastic material whose Lamé constants λ and μ satisfy $3\lambda + 2\mu > 0$ and $\mu > 0$.

In the *nonlinear* Koiter shell model, so named after Koiter [11], the *strain energy* associated with a sufficiently smooth deformation $\tilde{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ of the middle surface $\boldsymbol{\theta}(\overline{\omega})$ of a shell is defined by (the notations for the fundamental forms associated with $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are defined in Sect. 3)

$$\mathcal{E}(\tilde{\boldsymbol{\theta}}) := \frac{1}{2}\int_\omega \Big\{hW\big(\frac{1}{2}\boldsymbol{I}^{-1}(\tilde{\boldsymbol{I}} - \boldsymbol{I})\big) + \frac{h^3}{3}W\big(\boldsymbol{I}^{-1}(\tilde{\boldsymbol{II}} - \boldsymbol{II})\big)\Big\}\sqrt{\det \boldsymbol{I}}\,\mathrm{d}y$$

where the function $W : \mathbb{M}^2 \to \mathbb{R}$ is the *two-dimensional stored energy function* defined by

$$W(\boldsymbol{A}) := \frac{2\lambda\mu}{\lambda + 2\mu}(\operatorname{tr}\boldsymbol{A})^2 + 2\mu \operatorname{tr}\boldsymbol{A}^2 \text{ for each } \boldsymbol{A} \in \mathbb{M}^2.$$

It is easy to prove that there exist two constants $0 < C_1 \le C_2$, depending only on h, ω, and $\boldsymbol{\theta}$, such that

$$C_1\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)}\Big\} \le \mathcal{E}(\tilde{\boldsymbol{\theta}}) \le C_2\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)}\Big\}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,4}(\omega;\mathbb{R}^3)$ that satisfy $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,4}(\omega;\mathbb{R}^3)$.

Combined with the above estimate, the *nonlinear Korn inequalities on a surface* established in this paper *show that the strain energy* $\mathcal{E}(\tilde{\boldsymbol{\theta}})$ *"controls" the "magnitude" of the deformation of the middle surface of the shell*. In particular, Theorem 3.2(b) implies that there exists a constant C_3, which depends only on h, ω, and $\boldsymbol{\theta}$, such that

$$\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^4_{W^{1,4}(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|^4_{W^{1,4}(\omega)}\Big\} \le C_3\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)}\Big\} \le \frac{C_3}{C_1}\mathcal{E}(\tilde{\boldsymbol{\theta}})$$

and

$$\Big\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{H^1(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|^2_{H^1(\omega)}\Big\} \le C_3\Big\{\|\tilde{\boldsymbol{I}} - \boldsymbol{I}\|^2_{L^2(\omega)} + \|\tilde{\boldsymbol{II}} - \boldsymbol{II}\|^2_{L^2(\omega)}\Big\} \le \frac{C_3}{C_1}\mathcal{E}(\tilde{\boldsymbol{\theta}})$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,4}(\omega;\mathbb{R}^3)$ that satisfy $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) \in W^{1,4}(\omega;\mathbb{R}^3)$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0, $\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) = \boldsymbol{a}_3(\boldsymbol{\theta})$ on γ_0, and

$$|\tilde{R}_\alpha| \ge \varepsilon \text{ a.e. in } \omega,\ |\tilde{\boldsymbol{I}}| \le \frac{1}{\varepsilon} \text{ a.e. in } \omega, \text{ and } \big|\tilde{\boldsymbol{I}}^{-1}\big| \le \frac{1}{\varepsilon} \text{ a.e. in } \omega.$$

Note that a *formal linearization* of the latter nonlinear Korn inequality with respect to the displacement field

$$\boldsymbol{\zeta} := \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}$$

coincides with the following *linear Korn inequality on a surface*, due to Bernadou & Ciarlet [2] (see also [3,4]), which is the key to proving the existence and uniqueness of a solution to the *linear Koiter shell model* (in the strain energy of which the differences $(\tilde{\boldsymbol{I}} - \boldsymbol{I})$ and $(\tilde{\boldsymbol{II}} - \boldsymbol{II})$ are replaced by their linear parts with respect to $\boldsymbol{\zeta}$): There exists a constant C_4 depending only on h, ω, and $\boldsymbol{\theta}$, such that

$$\Big\{\sum_{\alpha=1}^{2}\|\zeta_\alpha\|^2_{H^1(\omega)} + \|\zeta_3\|^2_{H^2(\omega)}\Big\} \le C_4\Big\{\|\boldsymbol{\gamma}(\boldsymbol{\zeta})\|^2_{L^2(\omega)} + \|\boldsymbol{\rho}(\boldsymbol{\zeta})\|^2_{L^2(\omega)}\Big\},$$

for all vector fields $\boldsymbol{\zeta} : \omega \to \mathbb{R}^3$ whose components $\zeta_i := \boldsymbol{\zeta}\cdot\boldsymbol{a}_i(\boldsymbol{\theta}) : \omega \to \mathbb{R}$, $1 \le i \le 3$, satisfy

$$\zeta_\alpha \in H^1(\omega) \text{ and } \zeta_\alpha = 0 \text{ on } \gamma_0, \text{ and } \zeta_3 \in H^2(\omega) \text{ and } \zeta_3 = \partial_\alpha\zeta_3 = 0 \text{ on } \gamma_0,$$

where $\boldsymbol{\gamma}(\boldsymbol{\zeta})$, resp. $\boldsymbol{\rho}(\boldsymbol{\zeta})$, denotes the linear part with respect to $\boldsymbol{\zeta}$ of the difference $(\boldsymbol{I}(\boldsymbol{\theta}+\boldsymbol{\zeta}) - \boldsymbol{I}(\boldsymbol{\theta}))$, resp. of the difference $(\boldsymbol{II}(\boldsymbol{\theta}+\boldsymbol{\zeta}) - \boldsymbol{II}(\boldsymbol{\theta}))$.