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Ordinary differential equations/Partial differential equations

Almost automorphic evolution equations with compact almost automorphic solutions

Sur une classe d'équations d'évolution presque automorphes possédant des solutions compactes presque automorphes

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ABSTRACT

We prove that some almost automorphic evolution equations carry compact almost automorphic solutions. Moreover, we show that the almost automorphy of the coefficients is not necessary to obtain almost automorphic solutions. This improves the assumptions and the conclusion of a result of M. Zaki (Ann. Mat. Pura Appl. (4) 101 (1) (1974) 91–114), which gives the nature of solutions with relatively compact range for some almost automorphic evolution equations in Banach spaces. We note that many results in the literature can be improved in this direction.

RÉSUMÉ

Nous montrons que certaines équations d'évolution presque automorphes possèdent des solutions compactes presque automorphes. De plus, nous montrons que la presque automorphie des coefficients n'est pas nécessaire pour obtenir des solutions presque automorphes. Cela améliore les hypothèses et la conclusion d'un résultat de M. Zaki (Ann. Mat. Pura Appl. (4) 101 (1) (1974) 91–114), qui donne la nature des solutions avec image relativement compacte pour certaines équations d'évolution presque automorphes dans les espaces de Banach. Nous notons que de nombreux résultats dans la littérature peuvent être améliorés dans cette direction.

1. Introduction

In this work, we investigate the nature of solutions with relatively compact range for the following evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

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where A is the generator of a strongly asymptotically stable C_0-semigroup on a Banach space X and $f:\mathbb{R}\to X$ is an almost automorphic function in the sense of Stepanov.

Let us consider the following differential equation in $\mathbb{R}^n$:

$$x'(t) = G(t)x(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where the matrix $G(t)$ and the vector $f(t)$ are both continuous and ω-periodic for some $\omega > 0$. In [11], Massera proved that the existence of a bounded solution of Equation (1.2) on the positive real line is enough to get the existence of an ω-periodic solution. This result is known in the literature as the Massera theorem. Fixed-point theory plays an important role in this kind of results.

For almost periodic equations, the situation is more complicated, since one cannot use fixed-point arguments. Bohr and Neugebauer, see [8], extended Massera's theorem for Equation (1.2) to the almost periodic case when $G(t) = G$ is a constant matrix. In addition, they proved that a bounded solution of Equation (1.2) on $\mathbb{R}$ is automatically almost periodic. We note that this result does not hold for the periodic case.

In [5], Cooke proved that bounded solutions of the following differential equation

$$y^{(n)}(t) + A_1 y^{(n-1)}(t) + \ldots A_n y(t) = f(t),$$

are almost periodic when $f:\mathbb{R}\to H$ is almost periodic and A_i, $i = 1,\ldots,n$ are compact operators in a separable Hilbert space H. In [10], Haraux proved the same result for the following evolution equation

$$x'(t) + \widetilde{A}x(t) \ni f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

where $\widetilde{A}$ is a maximal monotone operator on $\mathbb{R}^2$. In [13,15], Zaidman proved that a bounded solution of Equation (1.1) is almost periodic when A is a self-adjoint operator in a Hilbert space. In another paper, Zaidman [16] proved this result for Equation (1.1) when A is a finite-rank operator. In [9], Goldstein extended the work of Zaidman by considering a more general finite dimensionality assumption when A is a closed linear operator in a Hilbert space.

Without some sort of finite dimensionality assumptions, one cannot predict in general that bounded solutions have a relatively compact range. In this case, it is more appropriate to look for almost periodic and almost automorphic solutions inside the set of solutions having a relatively compact range. Following this remark, Zaidman [14] showed that a bounded solution of Equation (1.1) with a relatively compact range is automatically almost periodic when f is also almost periodic and A generates a strongly asymptotically stable C_0-semigroup on a Banach space X. Under the same assumption on the operator A, Zaki [17] gave an analogous result for almost automorphic solutions. He proved that a bounded solution of Equation (1.1) with a relatively compact range is almost automorphic when f is also almost automorphic.

In this work, we strengthen the result of Zaki in the sense that we obtain a stronger conclusion and using weaker assumptions. More specifically, under the same strong asymptotic stability in [17], we prove that a bounded solution of Equation (1.1) with a relatively compact range is even compact almost automorphic when f is only almost automorphic in the sense of Stepanov. We note that many results in the literature can be improved in this direction.

2. Almost automorphic functions

Let $(X, |.|)$ be a Banach space and $BC(\mathbb{R}, X)$ be the space of bounded continuous functions from $\mathbb{R}$ to X equipped with the supremum norm

$$|f|_\infty := \sup_{t\in\mathbb{R}} |f(t)|. \tag{2.1}$$

In [4], Bochner introduced the concept of almost automorphy, which is a generalization of the almost periodicity.

Definition 2.1. [4] A continuous function $f:\mathbb{R}\mapsto X$ is said to be almost automorphic if for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$, such that for each $t \in \mathbb{R}$

$$f(t + s'_n) \to \widetilde{f}(t)$$

and

$$\widetilde{f}(t - s'_n) \to f(t)$$

as $n\to\infty$. If the above limits hold uniformly in compact subsets of $\mathbb{R}$, then f is said to be compact almost automorphic.

Remark. If one of the convergences in Definition 2.1 holds uniformly on the whole real line, then we obtain the notion of almost periodicity.