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Substitution-based structures with absolutely continuous spectrum

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Abstract

By generalising Rudin's construction of an aperiodic sequence, we derive new substitution-based structures which have a purely absolutely continuous diffraction measure and a mixed dynamical spectrum, with absolutely continuous and pure point parts. We discuss several examples, including a construction based on Fourier matrices which yields constant-length substitutions for any length.

Keywords: Substitution dynamical system; Spectral measure; Absolute continuity; Lebesgue spectrum

1. Introduction

Substitution dynamical systems are widely used as toy models for aperiodic order in one dimension [20,21]. By an argument of Dworkin [13], the diffraction spectrum of these systems is related to part of the dynamical spectrum, which is the spectrum of a unitary operator acting on a Hilbert space, as induced by the shift action. We refer the readers to [5] and references therein for recent developments and the current knowledge of the relationship between these different spectral characterisations. Here we are interested in systems that feature absolutely continuous spectra, in spite of being perfectly ordered.

A paradigm of such a system is the (binary) Rudin–Shapiro or Golay–Shapiro sequence.[1] It was introduced in [17,22,24] in answer to a question raised by Salem [22] in the context of

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[1] For simplicity, we will refer to this sequence as the Rudin–Shapiro sequence, as this is the more commonly used term.

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harmonic analysis; see also [4, Sec. 4.7.1] and [2]. This sequence, represented by a Dirac comb with balanced weights (± 1), is a substitution-based structure with purely absolutely continuous diffraction spectrum, so it has a mixed dynamical spectrum, with a pure point part arising from the underlying constant-length substitution structure. Indeed, this deterministic sequence has the stronger property that its two-point correlations vanish exactly for *any* non-zero distance; a direct proof of this property can be found in [4, Sec. 10.2]; see also [20]. Recall that the diffraction measure is the Fourier transform of the autocorrelation measure, which in this case is just δ_0 (a point measure located at the origin), so the diffraction measure is Lebesgue measure. Some generalisations of the Rudin–Shapiro sequence were provided in [1], but to date relatively few examples of substitution-based sequences of this type are known explicitly. There are good reasons for this, as one would expect a generic substitution-based structure to produce a singular continuous spectrum [6].

In [15], a systematic generalisation of the Rudin–Shapiro system to higher-dimensional substitutions was derived. It employs Hadamard matrices (matrices with elements ± 1 whose rows are mutually orthogonal). The underlying systems are symbolic constant-length substitutions on a finite alphabet $\mathcal{A}$, based on arrangements of letters on the (hyper)cubic lattice $\mathbb{Z}^d$. Letters in the alphabet are paired, so for each letter $a \in \mathcal{A}$ there is a twin letter $\overline{a} \in \mathcal{A}$, with $\overline{a} \neq a$ and $\overline{\overline{a}} = a$. In particular, the author proved the following result, where $\mathbb{X}$ denotes the hull of the substitution, μ the corresponding invariant measure, H_D the discrete spectrum, and $Z(f)$ the cyclic subspace associated to a function $f \in L^2(\mathbb{X}, \mu)$.

Theorem 1.1 (*[15]*). *Let* $(\mathbb{X}, \mathbb{Z}^d, \mu)$ *be a dynamical system associated to an aperiodic* $\mathbb{Z}^d$*-substitution subject to the conditions that*[2]

- *each letter in the alphabet* $\mathcal{A}$ *is only allowed to appear in the position given by its underlying number (so the images of letters under the substitution differ only in the number and/or position of the bars that distinguish paired letters);*
- *paired letters are substituted by corresponding paired blocks;*
- *the symbol matrix of the corresponding substitution is a Hadamard matrix.*

Then there exist functions $f_1, \dots, f_K \in L^2(\mathbb{X}, \mu)$*, each with spectral measure equal to Lebesgue measure, such that*

$$L^2(\mathbb{X}, \mu) = H_D \oplus Z(f_1) \oplus \cdots \oplus Z(f_K).$$

In this article we provide further examples of substitution-based structures with Lebesgue spectrum. These systems do not satisfy the last condition of Theorem 1.1, although they still appear to have a close relationship to Hadamard matrices and their complex analogues. Our approach is based on modifying and extending the original construction of Rudin [22]. As a consequence, our examples are sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ that satisfy the property

$$\sup_{|x|=1} \left| \sum_{n=1}^{N} \varepsilon_n x^n \right| \leq C\, N^{\frac{1}{2}} \tag{1}$$

for some positive constant C, where the supremum is taken over complex numbers of unit modulus. We shall refer to this property as the *root-N property*. This bound on the growth of the exponential sums implies that the corresponding diffraction measure is purely absolutely

[2] We refer to the original article for more details on these conditions.

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