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Algebraic sums and products of univoque bases

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Abstract

Given $x \in (0, 1]$, let $\mathcal{U}(x)$ be the set of bases $q \in (1, 2]$ for which there exists a unique sequence (d_i) of zeros and ones such that $x = \sum_{i=1}^{\infty} d_i/q^i$. Lü et al. (2014) proved that $\mathcal{U}(x)$ is a Lebesgue null set of full Hausdorff dimension. In this paper, we show that the algebraic sum $\mathcal{U}(x) + \lambda\mathcal{U}(x)$ and product $\mathcal{U}(x) \cdot \mathcal{U}(x)^{\lambda}$ contain an interval for all $x \in (0, 1]$ and $\lambda \neq 0$. As an application we show that the same phenomenon occurs for the set of non-matching parameters studied by the first author and Kalle (Dajani and Kalle, 2017).

Keywords: Algebraic differences; Non-integer base expansions; Univoque bases; Thickness; Cantor sets; Non-matching parameters

1. Introduction

Non-integer base expansions, a natural extension of dyadic expansions, have got much attention since the ground-breaking works of Rényi [18] and Parry [17]. Given a base $q \in (1, 2]$, an infinite sequence (d_i) of zeros and ones is called a *q-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i} =: ((d_i))_q.$$

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A number x has a q-expansion if and only if $x \in I_q := [0, \frac{1}{q-1}]$. Contrary to the dyadic expansions, Lebesgue almost every $x \in I_q$ has a continuum of q-expansions (see [19]). On the other hand, for each $k \in \mathbb{N} := \{1, 2, \ldots\}$ or $k = \aleph_0$ there exist $q \in (1, 2]$ and $x \in I_q$ such that x has precisely k different q-expansions (see [6]). For more information on the non-integer base expansions we refer to the survey paper [7] and the book chapter [3].

On the other hand, algebraic differences of Cantor sets and their connections with dynamical systems have been intensively investigated since the work of Newhouse [16], who introduced the notion of *thickness* to study whether a given Cantor set $C \subset \mathbb{R}$ has a non-empty intersection with its translations. Since $C \cap (C + t) \neq \emptyset$ if and only if $t \in C - C$, where the *algebraic difference* of two sets $A, B \subset \mathbb{R}$ is defined by $A - B := \{a - b : a \in A, b \in B\}$, the thickness (see Definition 3.1) can be used to study the algebraic difference of Cantor sets (cf. [1,13,14]).

In this paper, we consider the algebraic differences of sets of univoque bases for given real numbers. To be more precise, for $x \in (0, 1]$, let $\mathcal{U}(x)$ be the set of bases $q \in (1, 2]$ such that x has a unique q-expansion. Then each element of $\mathcal{U}(x)$ is called a *univoque base* of x. Lü et al. [15] proved that $\mathcal{U}(x)$ is a Lebesgue null set of full Hausdorff dimension.

We will prove the following result for the *algebraic sum* and *product* of $\mathcal{U}(x)$ defined respectively by

$$\mathcal{U}(x) + \lambda\mathcal{U}(x) := \{p + \lambda q : p, q \in \mathcal{U}(x)\} \quad \text{and} \quad \mathcal{U}(x) \cdot \mathcal{U}(x)^\lambda := \left\{pq^\lambda : p, q \in \mathcal{U}(x)\right\}.$$

Theorem 1.1. *For every $x \in (0, 1]$ and every $\lambda \neq 0$ both the sum $\mathcal{U}(x) + \lambda\mathcal{U}(x)$ and product $\mathcal{U}(x) \cdot \mathcal{U}(x)^\lambda$ contain an interval.*

We mention that the product $\mathcal{U}(x) \cdot \mathcal{U}(x)^\lambda$ in Theorem 1.1 can be converted to a sum by taking the logarithm and then repeating the construction (see Section 3 for more details). Hence, we will focus more on the algebraic sum $\mathcal{U}(x) + \lambda\mathcal{U}(x)$.

Remarks 1.2.

- *For $\lambda = -1$ Theorem 1.1 states that the algebraic difference $\mathcal{U}(x) - \mathcal{U}(x)$ and quotient $\mathcal{U}(x) \cdot \mathcal{U}(x)^{-1}$ contain an interval for each $x \in (0, 1]$.*
- *For $x = 1$ the set $\mathcal{U} := \mathcal{U}(1)$ is well-studied. For example, it has a smallest element $q_{KL} \approx 1.78723$, called the* Komornik–Loreti constant *(see [8]), and its closure $\overline{\mathcal{U}}$ is a Cantor set (see [9]). Furthermore, the local Hausdorff dimension of $\mathcal{U}$ is positive (see [12]), i.e.,* $\dim_H(\mathcal{U} \cap (q - \delta, q + \delta)) > 0$ *for any $q \in \mathcal{U}$ and $\delta > 0$. Theorem 1.1 for $x = 1$ and $\lambda = -1$ states that the algebraic difference $\mathcal{U} - \mathcal{U}$ and quotient $\mathcal{U} \cdot \mathcal{U}^{-1}$ contain an interval.*
- *The algebraic sum $\mathcal{U}(x) + \lambda\mathcal{U}(x)$ containing an interval for all $\lambda \neq 0$ can also be expressed by saying that for each $x \in (0, 1]$ and for each oblique straight line L passing through* 0*, the projection of the product set $\mathcal{U}(x) \times \mathcal{U}(x) = \{(p, q) : p, q \in \mathcal{U}(x)\}$ onto L contains an interval for all $x \in (0, 1]$.*

We will also show that the same phenomenon occurs for the set of non-matching parameters, recently studied by the first author and Kalle [2]. Let us introduce for each $\alpha \in [1, 2]$ the map $S_\alpha : [-1, 1] \to [-1, 1]$ by the formula

$$S_\alpha(x) = \begin{cases} 2x + \alpha, & \text{if} \quad -1 \le x < \frac{1}{2}, \\ 2x, & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2}, \\ 2x - \alpha, & \text{if} \quad \frac{1}{2} < x \le 1. \end{cases}$$

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