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Topological rigidity of linear cellular automaton shifts

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Abstract

We prove that topologically isomorphic linear cellular automaton shifts are algebraically isomorphic. Using this, we show that two distinct such shifts cannot be isomorphic. We conclude that the automorphism group of a linear cellular automaton shift is a finitely generated abelian group.

Keywords: Algebraic dynamics; Endomorphisms and automorphisms of compact abelian $\mathbb{Z}^d$ actions; Automorphism groups

1. Introduction

A full shift consists of a space, the set of doubly infinite sequences $\{1, 2, \ldots, q\}^{\mathbb{Z}}$, and the transformation σ acting on points in that space, defined by $\sigma(x)_n = x_{n+1}$. A multi-dimensional shift is defined in the same way. The space $\{1, 2, \ldots, q\}^{\mathbb{Z}^d}$ is acted on by d shifts, defined by

$$\sigma_i(x)_{(n_1,\ldots,n_i,\ldots,n_d)} = x_{(n_1,\ldots,n_i+1,\ldots,n_d)}$$

for transformations $\sigma_1, \sigma_2, \ldots, \sigma_d$. A *subshift* is a closed, shift-invariant subset of a full shift. It is *Markov* if it is a shift of finite type. We refer to [12] for definitions and the basic topological set-up.

In symbolic dynamics $\{1, \ldots, q\}$ is a finite set with no additional structure. In algebraic dynamics $\{1, \ldots, q\}$ is a finite abelian group or a finite field. In this paper, we limit ourselves to the simplest case, when q is prime. To emphasize this, we write p from now on, instead of q, and we denote the finite field by $\mathbb{F}_p$. Thus $\{1, 2, \ldots, p\}^{\mathbb{Z}^d}$ becomes a compact abelian group under coordinatewise addition $\mathbb{F}_p^{\mathbb{Z}^d}$. A *Markov subgroup* is a subshift that is also an additive subgroup

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https://doi.org/10.1016/j.indag.2018.05.011
0019-3577/

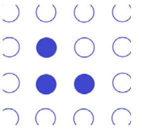
Please cite this article in press as: R. Fokkink, R. Yassawi, Topological rigidity of linear cellular automaton shifts, Indagationes Mathematicae (2018), https://doi.org/10.1016/j.indag.2018.05.011.

of $\mathbb{F}_p^{\mathbb{Z}^d}$. Any such group is a shift of finite type. A very nice survey of Markov groups is given in [6]. One motivation of our paper is to try and find topological analogues of the metric results in that survey.

The standard example of a Markov subgroup is the *Ledrappier shift* [11] defined by

$$\Lambda = \{(x)_{(m,n)} : x_{(m,n)} + x_{(m+1,n)} + x_{(m,n+1)} = 0 \text{ for each } i, j \in \mathbb{Z}^2\} \subset \{0, 1\}^{\mathbb{Z}^2}.$$

The defining relation of Λ corresponds to an L-shape in the lattice $\mathbb{Z}^2$:



It is convenient to identify $(c_{(i_1,\ldots,i_d)}) \in \mathbb{F}_p^{\mathbb{Z}^d}$ with the formal Laurent series

$$\sum_{(i_1,\ldots,i_d)\in\mathbb{Z}^d} c_{i_1,\ldots,i_d} X_1^{i_1} \cdots X_d^{i_d}.$$

The shift σ_i is given by the multiplication by X_i^{-1}. We denote the set of all Laurent series by $\mathbb{F}_p[[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]]$. A *Laurent polynomial* is a Laurent series in which all but finitely many coefficients are zero. We denote the set of Laurent polynomials by $\mathbb{F}_p[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. It is a unique factorization domain and the set of Laurent series $\mathbb{F}_p[[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]]$ is a module over this domain. A Markov subgroup is a subgroup of $\mathbb{F}_p[[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]]$ which is invariant under multiplication by X_i, i.e., it is a submodule. The annihilator of a Markov subgroup M is the ideal of all polynomials P such that $Px = 0$ for each $x \in M$. Annihilators are finitely generated since $\mathbb{F}_p[[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]]$ is Noetherian. We will be interested in particular in submodules $P^\perp$ that have an annihilator that is generated by a single polynomial P. For example, with this notation, the Ledrappier shift is equal to $\left(1 + X_1^{-1} + X_2^{-1}\right)^\perp$.

Two Markov shifts M_1 and M_2 are *isomorphic* if there exists an invertible map $\phi : M_1 \to M_2$ which is shift commuting, i.e., $\phi \circ \sigma_i = \sigma_i \circ \phi$ for all i. If ϕ is a homeomorphism, then it is a *topological isomorphism*. If ϕ is measure preserving, then it is a *measurable isomorphism*. If M_1 and M_2 are Markov subgroups and ϕ is an isomorphism between modules, then it is an *algebraic isomorphism*. An algebraic isomorphism is continuous and preserves the Haar probability measure, which is the only measure we consider. Kitchens conjectured that if $P^\perp$ and $Q^\perp$ are measurably isomorphic, then they are algebraically isomorphic [6]. This conjecture has been proved for irreducible and strongly mixing $P^\perp$ and $Q^\perp$:

Theorem 1 (*Kitchens–Schmidt, [10]*). *Suppose that P and Q are irreducible elements of $\mathbb{F}_p[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. If $P^\perp$ and $Q^\perp$ are measurably isomorphic and strongly mixing, then they are algebraically isomorphic.*

We are interested in the topological version of Kitchens's conjecture. We were unable to settle this topological version in full generality, and restrict our attention to polynomials of the following form: if

$$P = X_d - \Phi(X_1, \ldots, X_{d-1})$$

then we say that $P^\perp$ is a *linear cellular automaton shift*. Points of a linear cellular automaton shift, that is specific two dimensional configurations in these shifts, have been studied by many,

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