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Dynamical systems arising from random substitutions

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Abstract

Random substitutions are a natural generalisation of their classical 'deterministic' counterpart, whereby at every step of iterating the substitution, instead of replacing a letter with a predetermined word, every letter is independently replaced by a word from a finite set of possible words according to a probability distribution. We discuss the subshifts associated with such substitutions and explore the dynamical and ergodic properties of these systems in order to establish the groundwork for their systematic study. Among other results, we show under reasonable conditions that such systems are topologically transitive, have either empty or dense sets of periodic points, have dense sets of linearly repetitive elements, are rarely strictly ergodic, and have positive topological entropy.

Keywords: Primitive substitutions; Random substitutions; Symbolic dynamics

1. Introduction

Symbolic dynamical systems associated to primitive substitutions are the prototypical examples of minimal subshifts. As such, their study has been extensive [2,12,15,29] and various approaches to extending the theory have been explored, including *S-adic* or *mixed* systems [4,14,16,31], and systems associated to non-primitive substitutions [5,23]. Motivated by examples arising in physics within the study of quasicrystals, Godrèche and Luck considered the situation that the substituted image of a letter is a random variable [18], where we now call such systems *random* or *stochastic*. Others have independently studied similar generalisations of substitutions under the guise of *multi-valued* or *set-valued* substitutions [11], or *0L-systems* [30]. This randomised approach has recently been revisited [3,7,9,24,25,34] with several canonical examples now being established and studied (principally via their entropy and spectrum). In

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particular, Dekking has recently emphasised the need for a systematic approach in the study of random substitutions [10].

A general theory of random substitution subshifts (hereby abbreviated to RS-subshifts) has yet to be established. It is the goal of this article to remedy this situation and provide some key topological, dynamical and ergodic theoretic results with which one is usually accustomed when studying particular classes of symbolic subshifts. Throughout this work, we establish results which are direct generalisations of well-known results appearing in the classical study of deterministic substitutions. We highlight how the situation changes when moving from the deterministic to the random situation via examples, in order to illustrate the new and more interesting phenomena. Several of these results have been established previously for particular examples and we give particular mention to the Ph.D. thesis of Moll [24] from where several useful ideas have been borrowed.

In Section 2, we outline the basic definitions of random substitutions, their associated RS-subshifts, and introduce the primary standing assumption to be considered in this work; namely the *primitivity* of a substitution. Under assumption of primitivity, we establish a simple criterion in terms of the possible lengths of substituted letters for deciding when an RS-subshift is either empty or non-empty. We show that any element of the RS-subshift generates the entire RS-subshift as an orbit closure under the action of the shift and iterated substitution.

In Section 3 we establish the key dynamical and topological properties of an RS-subshift associated to a primitive random substitution. We prove that an RS-subshift is topologically transitive by constructing an explicit element with a dense shift-orbit. RS-subshifts are in general not minimal. We show a dichotomy result for the set of periodic points with respect to the shift: the set of periodic points is either empty or dense in an RS-subshift. Due to the potentially non-trivial structure of periodic points in these subshifts, this allows for robust tools such as the Artin–Mazur zeta function to be used in the study of random substitutions, unlike in the deterministic setting where the structure of periodic points is trivial. We show that, although RS-subshifts are in general not minimal, the set of minimal subspaces is dense in the subshift—in particular, we show that the set of linearly repetitive elements of an RS-subshift is dense. As a further dichotomy result, we show that an RS-subshift is either finite or is homeomorphic to a Cantor set. As a consequence of the topological transitivity of the subshift, we show that the associated tiling space is connected.

Section 4 is devoted to studying some measure-theoretic properties of RS-subshifts. A key tool used in establishing results is the notion of an *induced* or *collared* substitution. We expect that this will be a useful tool in the future study of random substitutions. The right Perron–Frobenius eigenvectors of the substitution matrices of these induced substitutions give rise to shift invariant (ergodic) measures. Moreover, we characterise those RS-subshifts which are uniquely/strictly ergodic.

In Section 5, we provide a very mild condition under which an RS-subshift exhibits positive topological entropy, together with loose lower bounds in terms of the letter-frequencies. This opens up the study of random substitutions to similar tools developed for the study of shifts of finite type and other positive entropy subshifts, where the topological entropy is a powerful invariant. Again, this is in contrast to the deterministic setting where the entropy is always zero.

We study several key examples in Section 6 which exhibit some of the more interesting behaviours described in the previous sections. We provide two very different representations of the full 2-shift as an RS-subshift. We show that the golden mean shift can also be realised as an RS-subshift—this leads to the question of whether SFTs can typically be represented as such. This question will be addressed in forthcoming work [19]. We show that examples of sofic shifts

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