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Indagationes Mathematicae ■ (■■■■) ■■■–■■■

indagationes mathematicae

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# The rigidity conjecture

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**Abstract**

A central question in dynamics is whether the topology of a system determines its geometry. This is known as rigidity. Under mild topological conditions rigidity holds for many classical cases, including: Kleinian groups, circle diffeomorphisms, unimodal interval maps, critical circle maps, and circle maps with a break point. More recent developments show that under similar topological conditions, rigidity does not hold for slightly more general systems. In this paper we state a conjecture which describes how topological classes are organized into rigidity classes.


*Keywords:* Rigidity; Renormalization; Smooth dynamics

## 1. Introduction

One of the aims of dynamics is to understand whether two dynamical systems are "topologically" the same. This is determined by the existence of a homeomorphism which conjugates the two systems. A related question is then to ask when two systems are "geometrically" the same. That is, when is the conjugacy differentiable?

This geometrical equivalence question has been studied in the last forty years in the case of circle diffeomorphisms, unimodal maps, critical circle maps, etc. (see Example 3.1). It turns out that, under mild topological restrictions, the conjugacy between two systems is differentiable as soon as it exists. In other words, the topology of a system determines its geometry. This is called the rigidity phenomenon.

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http://dx.doi.org/10.1016/j.indag.2017.08.001

One cannot expect the rigidity phenomenon in all generality. The mild topological restrictions are essential. For example, there is no rigidity in the context of circle diffeomorphisms when the rotation number is of strongly unbounded type, [1,2]. We will discuss the rigidity phenomenon only in the situation of bounded combinatorics. This is done with the purpose of stressing the fact that even in this simplest situation the rigidity phenomenon is more intricate than the classical case where "topology determines geometry".

Only in the last few years, further studies about the geometry of dynamical systems with bounded combinatorics have revealed classes for which the rigidity phenomenon does not hold. Non-rigidity occurs in natural classes of dynamical systems, such as: circle maps with a flat interval, Lorenz maps in one dimension and in Hénon maps in two dimensions. The geometrical equivalence of these systems is not solely determined by their topology. However, the rigidity phenomenon does not break down completely. Instead, the geometrical equivalence classes, called rigidity classes, are well organized inside the topological ones. The observed structures are

- foliations by rigidity classes,
- the coexistence phenomenon,
- probabilistic rigidity.

These notions are described in more detail in Sections 2 and 3.

The above examples and the structures that they revealed are what urged us to come up with a conjecture which describes the relation between the topological and geometric properties of a system. In Section 2 we discuss the resulting Rigidity Conjecture and in Section 3 we give examples supporting it.

## 2. The rigidity conjecture

In this section we present the basic notions needed to state the Rigidity Conjecture. The aim is to determine the geometry of the attractor of a system. The systems are smooth maps on manifold and the attractors are attractors in the sense of Milnor [3].

Two maps are in same **topological class** if they are conjugated on their attractors. Similarly, two maps are in same **rigidity class** if they are $C^{1+\alpha}$–conjugated on their attractors, for some $\alpha > 0$. A third notion of equivalence is given by so-called probabilistic rigidity. An attractor carries a dynamically relevant measure and we say that two maps are in the same **probabilistic rigidity class** if the conjugacy is $C^{1+\alpha}$ almost everywhere with respect to this measure, for some $\alpha > 0$. The topological class determines the topological properties of the attractor, whereas the rigidity class determines the attractor's geometrical properties.

We restrict our discussion to topological classes which are of **bounded combinatorics**. This topological property is well understood for one-dimensional systems and for infinitely renormalizable Hénon maps. For example, in case of circle diffeomorphisms bounded combinatorics is the same as saying that the rotation number is of bounded type. However, the topology of two and higher dimensional systems is still in the very beginning of its development. Part of the study of the rigidity phenomenon is to describe the topological restrictions needed for rigidity. At this moment our understanding of the topology of higher dimensional systems is too rudimentary to anticipate the general condition needed for rigidity. These topological restrictions will have the nature of being bounded.

Finally, a **stratification** of a topological class is a partition of the topological class into finite codimension submanifolds. The submanifolds can have different codimensions. Some of them can form a foliation.

Please cite this article in press as: M. Martens, et al., The rigidity conjecture, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.08.001.