ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae ▮ (▮▮▮▮) ▮▮▮–▮▮▮

indagationes mathematicae

www.elsevier.com/locate/indag

Error diffusion on simplices: The structure of bounded invariant tiles

R. Adler[1], T. Nowicki*, G. Świrszcz, C. Tresser, S. Winograd

IBM T.J. Watson Research Center, United States

To Sebastian van Strien on his 60th birthday

Abstract

This is a companion paper to Adleret al. (in press, 2015). There, we proved the existence of an absorbing invariant tile for the Error Diffusion dynamics on an acute simplex when the input is constant and "ergodic" and we discuss the geometry of this tile. Under the same assumptions we prove here that said invariant tile (a fundamental set of the lattice generated by the vertices of the simplex) which is a finite union of polytopes have the property that any union of the intersections of the tile with the Voronoï regions of the vertices is a tile for a different, explicitly defined lattice.

Keywords: Dynamical systems; Piece-wise translations; Error diffusion; Invariant sets

0. Simplices, lattices and error diffusion

Brief description of the results. This paper is one of the series of papers on absorbing invariant sets of piecewise (ergodic) translations on acute simplices. The existence of a bounded absorbing invariant set and the fact that it is a tile was proven in [6], where we also announced the results in the two next papers. Geometry, including the fact that this tile is a simplicial complex with the faces parallel to the faces of Voronoï regions and detailed analysis of the three dimensional case was described in [5]. Here we prove another dynamical property of the invariant tile, namely given a union of any sub-collection of Voronoï regions its intersection with the invariant tile is

* Corresponding author.

E-mail addresses: tnowicki@us.ibm.com (T. Nowicki), swirszcz@google.com (G. Świrszcz), charlestresser@yahoo.com (C. Tresser), swino@us.ibm.com (S. Winograd).

[1] Dr Roy Lee Adler, February 22, 1931–July 26, 2016.We are dearly missing you Roy.

https://doi.org/10.1016/j.indag.2017.10.001

Please cite this article in press as: R. Adler, et al., Error diffusion on simplices: The structure of bounded invariant tiles, Indagationes Mathematicae (2017), https://doi.org/10.1016/j.indag.2017.10.001.

again a tile for another, explicitly given lattice, which depends on the choice of the collection. This property, we hope, could lead to a renormalization theory for piecewise translations and to simultaneous Diophantine approximations.

Motivation. Piecewise continuous transformations are defined on a metric space equipped with a partition, by a continuous map acting on each element of the partition. Such transformations can serve for instance as models of regime switching in control theory [7], in scheduling [13,8], in game theory [12] or as continuous to discrete (and in particular analogue to digital) converters used in sigma-delta modulators [9] and color printing [2]. One can consider piecewise isometries such as rotations [10,3] or translations. An example of piecewise translations is a gradient descent for a piecewise affine function used in optimization and machine learning. For general motivation and background we refer to [6] and references therein.

We are interested here in the case of piecewise translations on an affine space $\mathbb{A}$ modeled on a d-dimensional real Euclidean vector space, with the metric defined by a scalar product. We investigate the case when the asymptotic behavior is bounded. One usually assumes some regularity of the partition, its elements are Borel sets with nonempty interiors or at least measurable sets of positive measure.

The $d+1$ translations in d-dimensional affine space. The simplest interesting case is when the partition consists of $d+1$ pieces $\mathbf{V}_i$ with $d+1$ translation vectors t_i, $i=0,\dots,d$. The map is defined by $\mathcal{F}(x)=x+t_i$ iff $x\in\mathbf{V}_i$. The vectors need to fulfill the condition that there exists a unique *positive convex combination* (*i.e.,* all the convex coefficients are strictly positive) of them producing 0: $\sum_{i=0}^{d}\gamma^i t_i=0$, $\sum_{i=0}^{d}\gamma^i=1, \gamma_i>0$. Any violation of this condition necessarily (for any partition) produces either unbounded trajectories or trajectories confined to a lower dimensional subspace. In particular for any j the collection $(t_i)_{i\neq j}$ is a basis of the vector space and the vector $-t_j$ lies in the *positive cone* generated by the remaining vectors: $-t_i\in\{\sum_{i\neq j}\lambda^i t_i, \lambda^i>0\}$.

The simplex, acuteness. Given a point $g\in\mathbb{A}$ such vectors produce a *simplex* $\triangle(v)$ in the affine space, with vertices $v_i=g-t_i$ and edges t_i-t_j. Because of the choice of the vectors t_i the points v_i are independent: they form a minimal set spanning the affine space $\mathbb{A}$. The interior of this simplex is given by all strictly positive convex combinations $\triangle^\circ(v)=\{\sum_{i=0}^{d}\xi^i v_i : \sum_{i=0}^{d}\xi^i=1, \xi^i>0\}$. Thus the point $g=\sum_{i=0}^{d}\gamma^i v_i$ lies in the interior of the simplex.

Such a simplex we call *acute* if all the *external normal vectors* s_j defined by $s_j\cdot(v_i-v_j)=1$ for all $i\neq j$ satisfy $s_i\cdot s_j<0$ for all $i\neq j$.

The lattice, the torus and the tile. The vertices of the simplex generate a discrete vector *lattice* $\mathbf{L}=\{\sum_{i=1}^{d}n_i(v_i-v_0), n_i\in\mathbb{Z}\}$ (note that the choice of the 0th index is arbitrary). After identifying points *equivalent* with respect to the lattice, where $x\equiv y$ iff $x-y\in\mathbf{L}$, we obtain the quotients space $\mathbb{A}/\mathbf{L}$ which is a d-dimensional *torus*. The map $\mathcal{F}$ can be projected onto a translation $[\mathcal{F}]$ on this torus, as all the translation vectors $t_i=g-v_i$ are projected onto the same vector in the quotient space.

A *tile* $\mathbf{T}$ of the lattice $\mathbf{L}$ is a subset of $\mathbb{A}$ which is projected bijectively onto the torus. For a set $\mathcal{R}$ of points in $\mathbb{A}$ and a set of vectors $\mathbf{K}$ define the map $\mathcal{R}+\mathbf{K}:\mathcal{R}\times\mathbf{K}\to\mathbb{A}$ by $(x,r)\mapsto x+r$. The set $\mathbf{T}$ is a tile for the lattice $\mathbf{L}$ iff $\mathbf{T}+\mathbf{L}$ is both *surjective* (each point of $\mathbb{A}$ is in the image) and *injective* (distinct arguments lead to distinct values). In particular for every point $x\in\mathbb{A}$ there exists a unique point of the tile $q\in\mathbf{T}$ and a unique vector of the lattice $r\in\mathbf{L}$ such that $x=q+r$. If a tile has some topological regularity then it is often called a fundamental region of the lattice.

Please cite this article in press as: R. Adler, et al., Error diffusion on simplices: The structure of bounded invariant tiles, Indagationes Mathematicae (2017), https://doi.org/10.1016/j.indag.2017.10.001.