ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae ▮ (▮▮▮▮) ▮▮▮–▮▮▮

indagationes mathematicae

www.elsevier.com/locate/indag

The affine separation problem revisited

Mihály Bessenyei

Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary

Received 31 October 2017; accepted 15 January 2018

Communicated by J. van Mill

Abstract

The aim of this note is to characterize in terms of inequalities those pairs of real functions (acting on a convex subset of a vector space) that possess an affine separator. The main result is originally due to Behrends and Nikodem. Their method is based on the Hahn–Banach Theorem and a variant of the Helly Theorem. In our approach, a direct and independent proof is presented via the Radon and the Helly Theorems.

Keywords: Sandwich theorem; Helly theorem; Radon theorem; Affine separation

1. Introduction

The classical Hahn–Banach Sandwich Theorem states that if a convex function majorizes a concave one on a convex subset of a vector space, then they can be separated by an affine function. This result can be considered as a *sufficient condition* for pairs of functions to have an affine separator. Therefore the question arises: Is it possible to find a *characteristic* property instead of the sufficient one? The same demand arises in connection of convex or concave separators.

Surprisingly, in both cases the answer is *positive*. The characterizations are given in terms of inequalities involving arbitrary long convex combinations. The theorem concerning the affine separation can be found in the book of Fuchssteiner and Lusky [10, p. 35], while the analogous result for convex separation is due to Baron, Matkowski and Nikodem [3]. Moreover, an abstract separation theorem can also be established: Besides further applications, the paper of Nikodem, Páles and Wąsowicz [12] involves all these cases.

E-mail address: besse@science.unideb.hu.

https://doi.org/10.1016/j.indag.2018.01.005

Separation problems come into the focus of Convex Geometry whence the underlying space is *finite dimensional*: It turns out that the length of the characteristic inequalities can uniformly be reduced "close" to the dimension of the domain. Using the Carathéodory Theorem [8], the problem of convex separation is solved by Baron, Matkowski and Nikodem [3]. The affine correspondence is due to Nikodem and Wąsowicz [13] provided that the vector space is the real line. Their method relies on the Santaló Theorem [15]. The general case is answered by Behrends and Nikodem [5]. In fact, first they prove an affine selection theorem via a variant of the Helly Theorem [11], and then they combine it with the Hahn–Banach Theorem. Let us recall here their obtained corollary:

Theorem. *Let D be a convex subset of a d-dimensional vector space X. There exists an affine separator between the functions $f, g : D \to \mathbb{R}$ if and only if*

$$\sum_{k=1}^{n} \lambda_k f(x_k) \le \sum_{l=1}^{m} \mu_l g(y_l) \tag{1}$$

holds for all convex combinations $\lambda_1 x_1 + \cdots + \lambda_n x_n = \mu_1 y_1 + \cdots + \mu_m y_m$ with base points in D and with $n + m = \dim(X) + 2$.

Applying a result of Bárány [2], a further generalization of this statement is presented by Balaj and Nikodem [1]. The aim of our paper is to revisit the above theorem of Behrends and Nikodem with an alternative, elementary and self-contained approach. This direct way completely avoids the use of set-valued mappings. Besides the Helly Theorem, our key tool is the Radon Theorem [14]. Some ideas from the papers [6] and [7] are also adopted.

2. Proof of the affine separation theorem

In the forthcomings, the Helly Theorem [11] and the Radon Theorem [14] play a crucial role. For their detailed form, consult the book of Barvinok [4] or the monograph of Danzer, Grünbaum and Klee [9]. The abbreviations Conv, Aff, Lin will stand for the convex, affine and linear hull operators. The set of real valued affine functions acting on a vector space X will be denoted by $\mathscr{A}(X, \mathbb{R})$. The first two technical lemmas subsume the most important properties of $\mathscr{A}(X, \mathbb{R})$ and its certain subsets. The third lemma is a dimension formula for intersecting affine hulls.

Lemma 1. *Let $\{p_1, \ldots, p_{d+1}\}$ be an affine independent subset of the d-dimensional vector space X. Then,*

$$\|h\|_\infty := \max\{|h(p_1)|, \ldots, |h(p_{d+1})|\}$$

defines a norm on $\mathscr{A}(X, \mathbb{R})$.

Hint. Most of the norm properties can be checked quite easily. The only one which needs some explanation is that $\|h\| = 0$ implies $h = 0$. However, this follows from the fact that any affine function is determined uniquely by its values at a maximal affine independent set. □

Lemma 2. *If X is a d-dimensional vector space, $f, g : D \to \mathbb{R}$ are given functions fulfilling $f \le g$, then the set*

$$H(p) := \{h \in \mathscr{A}(X, \mathbb{R}) \mid f(p) \le h(p) \le g(p)\}$$

is a nonempty, convex and closed subset of $\mathscr{A}(X, \mathbb{R})$ for any $p \in D$. Moreover, if $\{p_1, \ldots, p_{d+1}\}$ is an affine independent subset of X, then $H := H(p_1) \cap \ldots \cap H(p_{d+1})$ is compact.