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Characterization of distributivity in a solid

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Abstract

We give a characterization of the validity of the distributive law in a solid. There exists equivalence between the characterization and the modified axiom of distributivity valid in a solid.

Keywords: Solids; Distributivity

1. Introduction

Solids arise as extensions of fields [2], typically non-Archimedean fields or the nonstandard reals [4,5], in the form of cosets with respect to convex subgroups. Such convex subgroups may be seen as orders of magnitude and are called *magnitudes*. In solids the laws of addition and multiplication are more those of completely regular semigroups [3,6] than of proper groups. Also the distributive law is not valid in general, but there exists an adapted form of distributivity, introducing a correction term in the form of a magnitude. In this paper we characterize the validity of the ordinary distributive law in a solid (Theorem 4.2). Let x, y, z be arbitrary elements in a solid. The conditions of the characterization given in this paper roughly state that in order for distributivity to fail the factor x should be more imprecise than the terms y and z, and these terms should be almost opposite. Special cases where distributivity does hold include magnitudes, elements of the same sign, and precise elements (elements with minimal magnitude).

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The equality expressed by the adapted distributivity axiom of [2] (Axiom 22 of the Appendix) and the characterization of distributivity by Theorem 4.2 are shown to be equivalent.

The results require a thorough investigation of the properties of magnitudes and precise elements. This is done in Sections 2 and 3. In Section 4.1 we state necessary and sufficient conditions for distributivity to hold. In Section 4.2 we prove that the equivalency of these conditions to proper distributivity is equivalent to the distributivity law with correction term, as given by Axiom 22.

For notation and terminology we refer to [2]. For the sake of reference a complete list of axioms is given in Appendix. We recall that, given an element x of a solid, its individualized neutral element e (see Axiom 3) as such is unique, and has the functional notation $e = e(x)$. In the same way the symmetric element s of Axiom 3 is denoted by $s(x) \equiv -x$, the individualized unity u of Axiom 8 is denoted by $u(x)$ and the multiplicative inverse d of Axiom 9 is denoted by $d(x) \equiv x^{-1} \equiv 1/x$.

2. Algebraic properties of magnitudes

In this section we study algebraic properties of magnitudes in an assembly. Assemblies were introduced in [1]. The results in this section will be used to prove the characterization of distributivity in a solid.

2.1. Neutral and symmetric elements

We verify some elementary properties of magnitudes and symmetrical elements. Part of it are generalizations of the usual properties of neutral and symmetric elements and others deal with their functional representation.

We start by recalling some results on magnitudes of [1].

Theorem 2.1 (*[1, Thm 4.11]*). (Cancellation law) *Let A be an assembly and let $x, y, z \in A$. Then $x + y = x + z$ if and only if $e(x) + y = e(x) + z$.*

Proposition 2.2 (*[1, Thm 4.12]*). *Let A be an assembly. Then for all $x, y \in A$*

(1) (Idempotency for addition) $e(x) + e(x) = e(x)$.
(2) (Linearity of e) $e(x + y) = e(x) + e(y)$.
(3) (Absorption) $e(x) + e(y) = e(x)$ or $e(x) + e(y) = e(y)$.
(4) (Idempotency for composition) Let A be an assembly and let $x \in A$. Then $e(e(x)) = e(x)$.

As a consequence a magnitude can only be the magnitude of itself.

Theorem 2.3 (*Representation*). *Let A be an assembly and let $x, y \in A$. If $x = e(y)$ then $x = e(x)$.*

Proof. Suppose $y \in A$ is such that $x = e(y)$. Then $e(x) = e(e(y)) = e(y) = x$ by Proposition 2.2.4. □

As regards to the symmetric function, it is easy to see that it is injective. We verify now that it is linear with respect to addition and has the symmetry property, meaning that the inverse of the inverse of a given element is the element itself. We also show that the composition of the inverse function with the neutral function is equal to the neutral function.

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