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The quartet spaces of G. ’t Hooft

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Abstract

In 1964, G. ’t Hooft postulated three axioms, and proved that every nonempty finite model of them has 4^n elements. This note confirms this by showing that every nonempty model can be made into a vector space over the field with four elements. For every pair of different elements x and y, the quartet of x and y is the affine line through x and y in this vector space.

Keywords: Axiom system; Finite field; Vector space

1. Introduction

In 1964, when I was an undergraduate in physics at Utrecht University in the Netherlands, the now famous physicist Gerard ’t Hooft was one of us. When our professor Freudenthal taught us the axioms of linear spaces, ’t Hooft postulated an axiom system out of the blue. He called the models of his axiom system quartet spaces because they are compositions of quartets, models with four elements. Indeed, he proved that every nonempty finite model has 4^n elements for some natural number n, and that such models exist for all n.

He never published this. Around 1966, I wrote about his quartet spaces in our undergraduate newsletter WisFysVaria. In the present note I present his axiom system and show that every nonempty model of it is in some sense a vector space over the field with four elements.

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2. Axioms for quartet spaces

Let a $*$-space be a set Q with a binary operation $*$ between its elements. 't Hooft defined a *quartet space* Q to be a $*$-space Q that satisfies the axioms

(A0) $x * y = z$ implies $y * z = x$,
(A1) $x * (y * z) = z * (y * x)$,
(A2) $x * x = x$,

for all $x, y, z \in Q$.

Recall that a $*$-space Q is called commutative and associative if it satisfies the respective axioms

(com) $x * y = y * x$,
(ass) $x * (y * z) = (x * y) * z$.

In general, a quartet space is neither commutative nor associative. Therefore, we have to be careful with the order and the association of the operands.

If Axiom (A0) is applied twice, it yields

$$x * y = z \quad \text{implies} \quad z * x = y. \tag{0}$$

Elimination of z from the implications (A0) and (0) gives the equalities

$$y * (x * y) = x \quad \text{and} \quad (x * y) * x = y. \tag{1}$$

Axiom (A1) has a mirror version

$$(x * y) * z = (z * y) * x. \tag{2}$$

This formula follows from the axioms (A0) and (A1) because

$$\begin{array}{ll} & (x * y) * z \\ = & \{ \text{ take } b = z * y, \text{ so that } z = y * b \text{ by (A0) } \} \\ & (x * y) * (y * b) \\ = & \{ \text{ (A1) with } x := x * y \text{ and } z := b \ \} \\ & b * (y * (x * y)) \\ = & \{ \text{ value of } b, \text{ and (1) with } x \text{ and } y \text{ interchanged } \} \\ & (z * y) * x. \end{array}$$

A less obvious consequence is:

$$(w * x) * (y * z) = (w * y) * (x * z). \tag{3}$$

Formula (3) follows from Axiom (A1) because

$$\begin{array}{ll} & (w * x) * (y * z) \\ = & \{ \text{ (A1) with } x := w * x \} \\ & z * (y * (w * x)) \\ = & \{ \text{ (A1) with } x := y, \ y := w, \ z := x \} \\ & z * (x * (w * y)) \\ = & \{ \text{ (A1) with } x := z, \ y := x, \ z := w * y \} \\ & (w * y) * (x * z). \end{array}$$

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