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Compact-like operators in lattice-normed spaces

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Abstract

A linear operator T between two lattice-normed spaces is said to be p-compact if, for any p-bounded net x_α, the net Tx_α has a p-convergent subnet. p-Compact operators generalize several known classes of operators such as compact, weakly compact, order weakly compact, AM-compact operators, etc. Similar to M-weakly and L-weakly compact operators, we define p-M-weakly and p-L-weakly compact operators and study some of their properties. We also study up-continuous and up-compact operators between lattice-normed vector lattices.

Keywords: Compact operator; Vector lattice; Lattice-normed space; Lattice-normed vector lattice; up-convergence; Mixed-normed space

1. Introduction

It is known that order convergence in vector lattices is not topological in general. Nevertheless, via order convergence, continuous-like operators (namely, order continuous operators) can

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be defined in vector lattices without using any topological structure. On the other hand, compact operators play an important role in functional analysis. Our aim in this paper is to introduce and study compact-like operators in lattice-normed spaces and in lattice-normed vector lattices by developing topology-free techniques.

Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice X is order convergent (or o-convergent, for short) to $x \in X$, if there exists another net $(y_\beta)_{\beta\in B}$ satisfying $y_\beta \downarrow 0$, and for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. In this case we write $x_\alpha \xrightarrow{o} x$. In a vector lattice X, a net x_α is unbounded order convergent (or uo-convergent, for short) to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$; see [11]. In this case we write $x_\alpha \xrightarrow{uo} x$. In a normed lattice $(X, \|\cdot\|)$, a net x_α is unbounded norm convergent to $x \in X$, written as $x_\alpha \xrightarrow{un} x$, if $\| |x_\alpha - x| \wedge u\| \to 0$ for every $u \in X_+$; see [8]. Clearly, if the norm is order continuous then uo-convergence implies un-convergence. Throughout the paper, all vector lattices are assumed to be real and Archimedean.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a vector norm (i.e. $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$) then the triple (X, p, E) is called a *lattice-normed space*, abbreviated as LNS. The lattice norm p in an LNS (X, p, E) is said to be *decomposable* if for all $x \in X$ and $e_1, e_2 \in E_+$, it follows from $p(x) = e_1 + e_2$, that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. If X is a vector lattice, and the vector norm p is monotone (i.e. $|x| \le |y| \Rightarrow p(x) \le p(y)$) then the triple (X, p, E) is called a *lattice-normed vector lattice*, abbreviated as LNVL. In this article we usually use the pair (X, E) or just X to refer to an LNS (X, p, E) if there is no confusion.

We abbreviate the convergence $p(x_\alpha - x) \xrightarrow{o} 0$ as $x_\alpha \xrightarrow{p} x$ and say in this case that x_α p-converges to x. A net $(x_\alpha)_{\alpha\in A}$ in an LNS (X, p, E) is said to be *p-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ p-converges to 0. An LNS (X, p, E) is called (*sequentially*) *p-complete* if every p-Cauchy (sequence) net in X is p-convergent. In an LNS (X, p, E) a subset A of X is called *p-bounded* if there exists $e \in E$ such that $p(a) \le e$ for all $a \in A$. An LNVL (X, p, E) is called *op-continuous* if $x_\alpha \xrightarrow{o} 0$ implies that $p(x_\alpha) \xrightarrow{o} 0$.

A net x_α in an LNVL (X, p, E) is said to be *unbounded p-convergent* to $x \in X$ (shortly, x_α up-converges to x or $x_\alpha \xrightarrow{up} x$), if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ for all $u \in X_+$; see [4, Def.6].

Let (X, p, E) be an LNS and $(E, \|\cdot\|_E)$ be a normed lattice. The *mixed norm* on X is defined by $p\text{-}\|x\|_E = \|p(x)\|_E$ for all $x \in X$. In this case the normed space $(X, p\text{-}\|\cdot\|_E)$ is called a *mixed-normed space* (see, for example [13, 7.1.1, p.292]).

A net x_α in an LNS (X, p, E) is said to *relatively uniformly p-converge* to $x \in X$ (written as, $x_\alpha \xrightarrow{rp} x$) if there is $e \in E_+$ such that for any $\varepsilon > 0$, there is α_ε satisfying $p(x_\alpha - x) \le \varepsilon e$ for all $\alpha \ge \alpha_\varepsilon$. In this case we say that x_α rp-converges to x. A net x_α in an LNS (X, p, E) is called *rp-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ rp-converges to 0. It is easy to see that for a sequence x_n in an LNS (X, p, E), $x_n \xrightarrow{rp} x$ iff there exist $e \in E_+$ and a numerical sequence $\varepsilon_k \downarrow 0$ such that for all $k \in \mathbb{N}$ and there is $n_k \in \mathbb{N}$ satisfying $p(x_n - x) \le \varepsilon_k e$ for all $n \ge n_k$. An LNS (X, p, E) is said to be *rp-complete* if every rp-Cauchy sequence in X is rp-convergent. It should be noticed that in a rp-complete LNS every rp-Cauchy net is rp-convergent. Indeed, assume x_α is a rp-Cauchy net in a rp-complete LNS (X, p, E). Then an element $e \in E_+$ exists such that, for all $n \in \mathbb{N}$, there is an α_n such that $p(x_{\alpha'} - x_\alpha) \le \frac{1}{n} e$ for all $\alpha, \alpha' \ge \alpha_n$. We select a strictly increasing sequence α_n. Then it is clear that x_{α_n} is rp-Cauchy sequence, and so there is $x \in X$ such that $x_{\alpha_n} \xrightarrow{rp} x$. Let $n_0 \in \mathbb{N}$. Hence, there is α_{n_0} such that for all $\alpha \ge \alpha_{n_0}$ we have $p(x_\alpha - x_{\alpha_{n_0}}) \le \frac{1}{n_0} e$ and, for all $n \ge n_0$ $p(x - x_{\alpha_{n_0}}) \le \frac{1}{n_0} e$, from which it follows that $x_\alpha \xrightarrow{rp} x$.

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