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Additivity, subadditivity and linearity: Automatic continuity and quantifier weakening

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Abstract

We study the interplay between additivity (as in the Cauchy functional equation), subadditivity and linearity. We obtain automatic continuity results in which additive or subadditive functions, under minimal regularity conditions, are continuous and so linear. We apply our results in the context of quantifier weakening in the theory of regular variation, completing our programme of reducing the number of hard proofs there to zero.

Keywords: Subadditive; Sublinear; Shift-compact; Analytic spanning set; Additive subgroup; Hamel basis; Steinhaus sum theorem; Heiberg–Seneta conditions; Thinning; Regular variation

1. Introduction

Our main theme here is the interplay between additivity (as in the Cauchy functional equation), subadditivity and linearity. As is well known, in the presence of smoothness conditions (such as continuity), additive functions $A : \mathbb{R} \to \mathbb{R}$ are linear, so of the form $A(x) = cx$. There is much scope for *weakening* the smoothness requirement and also much scope for weakening

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the universal quantifier by *thinning* its range $\mathbb{A}$ below from the classical context $\mathbb{A} = \mathbb{R}$:

$$A(u+v) = A(u) + A(v)\ (\forall u, v \in \mathbb{A}). \qquad (\mathrm{Add}_{\mathbb{A}}(A))$$

We address the Cauchy functional equation in Section 2. The philosophy behind our quantifier weakening[1] theorems is to establish *linearity* of a function F on $\mathbb{R}$ from its *additivity* on a thinner set $\mathbb{A}$ and from additional ('side') conditions, which include its *extendability* to a subadditive function; recall that $S : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *subadditive* [40, Ch. 3] if for all $u, v \in \mathbb{R}$

$$S(u+v) \leqslant S(u) + S(v), \qquad (\mathrm{Sub})$$

whenever meaningful on the right-hand side (cf. [50, 16.1] for subadditive and [64, p. 23 ffol] for convex functions); see also [53], [10]. (This choice for the setting is more convenient than alternatively working on $\mathbb{R}_+$, even though one-sided side-conditions are important here.) To motivate our main result we begin with an automatic continuity theorem, devoted entirely to subadditive functions; it implies a result about those linear functions that have subadditive extensions—see Proposition 7 (on uniqueness of extension). This (Theorem 0) makes explicit an argument springing from a step in a proof by Goldie, of Th. 3.2.5 in [8] (BGT below, for brevity), recently improved and generalized in [22] (though still implicit even there).

We recall that for S subadditive and finite-valued, $S(0) \geqslant 0$, as $S(0) \leqslant S(0) + S(0)$, so that $S(0) = 0$ iff $S(-z) = -S(z)$ for some z, as will be the case below when S extends an additive function; cf. [50, p. 457].

Theorem 0. *For subadditive $S : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ with $S(0+) = S(0) = 0$: S is continuous at 0 iff $S(z_n) \to 0$, for some sequence $z_n \uparrow 0$, and then S is continuous everywhere, if finite-valued.*

The last part above draws on [40, Th. 2.5.2] that, for a subadditive function, continuity at the origin implies continuity everywhere. Theorem 0, in the presence of right-sided continuity, asserts that the merest hint of left-sided continuity gives full continuity; contrast this with the behaviour of the subadditive function $\mathbf{1}_{[0,\infty)}$, which is continuous on the right but not on the left. This leads to the question of whether right-sided continuity can be thinned out. We are able to do so in the next two results below, but at the cost of imposing more structure, either on the left, or on the right. We need the following two definitions.

Definitions. 1. Say that Σ is *locally Steinhaus–Weil (SW)*, or has the *SW property locally*, if for $x, y \in \Sigma$ and, for all $\delta > 0$ sufficiently small, the sets

$$\Sigma_z^\delta := \Sigma \cap B_\delta(z),$$

for $z = x, y$, have the *interior-point property*, that $\Sigma_x^\delta \pm \Sigma_y^\delta$ has $x \pm y$ in its interior. (Here $B_\delta(x)$ is the open ball about x of radius δ.) See [18] for conditions under which this property is implied by the interior-point property of the sets $\Sigma_x^\delta - \Sigma_x^\delta$ (cf. [3]); for a rich list of examples, see Section 4.
2. Say that $\Sigma \subseteq \mathbb{R}$ is *shift-compact* if for each *null sequence* $\{z_n\}$ (i.e. with $z_n \to 0$) there are $t \in \Sigma$ and an infinite $\mathbb{M} \subseteq \mathbb{N}$ such that

$$\{t + z_m : m \in \mathbb{M}\} \subseteq \Sigma.$$

See [19], and for the group-action aspects, [57].

[1] Or 'quantifier easing'.

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