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Spectral radius of power graphs on certain finite groups

Sriparna Chattopadhyay[a,*], Pratima Panigrahi[b], Fouzul Atik[b]

[a] School of Mathematical Sciences, National Institute of Science Education and Research, Bhubaneswar, India
[b] Department of Mathematics, Indian Institute of Technology, Kharagpur, India

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Abstract

The power graph of a group G is a graph with vertex set G and two distinct vertices are adjacent if and only if one is an integral power of the other. In this paper we find both upper and lower bounds for the spectral radius of power graph of cyclic group C_n and characterize graphs for which these bounds are extremal. Further we compute spectra of power graphs of dihedral group D_{2n} and dicyclic group Q_{4n} partially and give bounds for the spectral radii of these graphs.

Keywords: Finite group; Power graph; Graph spectrum; Spectral radius

1. Introduction

The *power graph* $\mathcal{G}(G)$ of a group G is an undirected graph whose vertex set is G and two vertices $u, v \in G$ are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m.

The concept of (directed) power graphs related to semigroups and groups was introduced by Kelarev and Quinn [11,12]. Subsequently Chakrabarty et al. [5] defined the power graph of a semigroup and characterized the class of semigroups S for which $\mathcal{G}(S)$ is connected and complete. As a consequence they proved that for any finite group G, $\mathcal{G}(G)$ is always connected and is complete if and only if G is a cyclic group of order 1 or p^m, for some prime p and positive

* Corresponding author.
E-mail addresses: sriparnamath@gmail.com (S. Chattopadhyay), pratima@maths.iitkgp.ernet.in (P. Panigrahi), fouzulatik@gmail.com (F. Atik).

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integer m. After that the undirected power graph became the main focus of study in [4,6,9]. In [4] Cameron proved that for a finite cyclic group C_n of non-prime-power order n, the set of vertices Y_n of $\mathcal{G}(C_n)$ which are adjacent to all other vertices of $\mathcal{G}(C_n)$ consists of the identity and generators of C_n only. So $|Y_n| = 1 + \phi(n)$, where $\phi(n)$ is the Euler's ϕ function. For more on power graphs, we refer the reader to the survey paper [1] and the references therein.

Recall that for any finite simple graph $\mathbb{G}$ with vertex set $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(\mathbb{G}) = (a_{ij})$ is defined as an $n \times n$ matrix, where $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise. The (adjacency) characteristic polynomial of a graph $\mathbb{G}$ is given by $P(\mathbb{G}, x) = det(xI - A(\mathbb{G}))$. The eigenvalues of $A(\mathbb{G})$ are called the eigenvalues of $\mathbb{G}$ and denoted by $\lambda_i(\mathbb{G})$, $i = 1, 2, \ldots, n$. Clearly $A(\mathbb{G})$ is a symmetric matrix and so all its eigenvalues are real. Thus they can be arranged in non increasing order as $\lambda_1(\mathbb{G}) \geq \lambda_2(\mathbb{G}) \geq \cdots \geq \lambda_n(\mathbb{G})$. The multiset of all eigenvalues of $\mathbb{G}$ is called the *spectrum* of $\mathbb{G}$ and the largest eigenvalue $\lambda_1(\mathbb{G})$ is called the *spectral radius* of $\mathbb{G}$. Spectra of graphs have wide applications in quantum chemistry whereas spectral radius plays an important role in modelling of virus propagation in computer networks. Also eigenvalues of the adjacency matrix (in particular, the spectral radius) are useful tools to protect the privacy of personal data in some databases. For more than four decades, computing the spectrum of algebraic graphs is an interesting research field, see for example [2,10].

Throughout the paper we will denote the cyclic group of order n as C_n, dihedral group of order $2n$ as D_{2n}, and dicyclic group of order $4n$ by Q_{4n}. It is well known that C_n is isomorphic to the additive group $\mathbb{Z}_n$. Chattopadhyay and Panigrahi [7] computed Laplacian spectra of power graphs of finite cyclic and dihedral groups. Recently Mehranian et al. [13] calculated the characteristic polynomials of the power graphs of cyclic groups, elementary abelian groups of prime power order and dihedral groups whose order is twice a prime power. In this paper we determine characteristic polynomials of the power graphs of dihedral groups D_{2n} for any integer $n \geq 3$, and also of the generalized quaternion 2-groups. However from the characteristic polynomials, it is often difficult to compute the eigenvalues of $\mathcal{G}(\mathbb{Z}_n)$, $\mathcal{G}(D_{2n})$ and $\mathcal{G}(Q_{4n})$ whenever n has a large number of factors. In this type of situation, it is a general practice to obtain bounds for eigenvalues, specially bounds for spectral radius. For instance, one may see the book [3] and the references therein. Hence we present both upper and lower bounds for spectral radii of power graphs of finite cyclic, dihedral and dicyclic groups.

2. Main results

In [13] the characteristic polynomial of $\mathcal{G}(\mathbb{Z}_n)$ has been obtained in terms of the characteristic polynomial of the quotient matrix T whose entries are some functions of the divisors of n. Also note that the spectral radius of $\mathcal{G}(\mathbb{Z}_n)$ is same as that of the matrix T. Since the increase in the number of factors of n leads to a rapid increase of the degree of the characteristic polynomial of T it is sometimes too complicated to find the exact value of the spectral radius of $\mathcal{G}(\mathbb{Z}_n)$. Therefore one can use some graph invariants like vertex degrees, diameter etc. to approximate the spectral radius. The following theorem gives both upper and lower bounds on the spectral radius of $\mathcal{G}(C_n)$ in terms of the maximum and minimum degree of the non-identity non-generator elements of C_n.

Theorem 2.1. *For any natural number $n \geq 3$ the spectral radius $\lambda_1(\mathcal{G}(C_n))$ of $\mathcal{G}(C_n)$ satisfies the following.*

$$\lambda_1(\mathcal{G}(C_n)) \geq \frac{1}{2}\left[(d_{min} - 1) + \sqrt{(2l - 1 - d_{min})^2 + 4l(n - l)}\,\right]$$